TRIANGULATED CATEGORIES AND REPRESENTATION THEORY

\# 2024-09-25

Let $\mathscr{C}$ be a category and S a multiplicative system of $\mathscr{C}$. The localization of $\mathscr{C}$ by S, denoted $\mathscr{C}_S$, has as hom-sets

$$\mathrm{Hom}_{\mathscr{C}_S}(X,Y) = \left\{ (X',s,f) \;\middle|\; \begin{array}{l} X' \in \mathrm{Ob}\,\mathscr{C},\ s \in S \\ s \in \mathrm{Hom}_{\mathscr{C}}(X',X), \\ f \in \mathrm{Hom}_{\mathscr{C}}(X',Y) \end{array} \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation defined below.

<u>Notation.</u> A triple (X',s,f) is drawn as the roof

$$\begin{tikzcd} & X' \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

and we denote it $\frac{f}{s}$. (The $\sim$ denotes $s \in S$)

<u>Def.</u> Given (X',s,f) and (X'',t,g), we define $\frac{f}{s} \sim \frac{g}{t}$ iff there is $u \in S$, $\alpha, \beta \in \mathscr{C}$ s.t. the following diagram commutes

$$\begin{tikzcd} & X & \\ X' \arrow[ur, "s"] \arrow[dr, "f"'] & X''' \arrow[u, "u"] \arrow[l, "\alpha"] \arrow[r, "\beta"'] & X'' \arrow[ul, "t"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

The equivalence class of the roof (X',s,f) is denoted $\left[\frac{f}{s}\right]$.

<u>GOAL</u> To show that composition of classes of roofs is well-defined.

In other words: $\frac{f}{s} \sim \frac{f'}{s'}$ and $\frac{g}{t} \sim \frac{g'}{t'} \Rightarrow \frac{g}{t} \circ \frac{f}{s} \sim \frac{g'}{t'} \circ \frac{f'}{s'}$
for $\left[\frac{f}{s}\right] \in \mathrm{Hom}_{\mathscr{C}_S}(X,Y)$ and $\left[\frac{g}{t}\right] \in \mathrm{Hom}(Y,Z)$.

Def of Composition Fix two roofs
$$\begin{tikzcd} & X' \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y' \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ Y & & Z \end{tikzcd}.$$

If there exist a pair of arrows
$$X' \xleftarrow[\sim]{s'} W \xrightarrow{t'} Y'$$
s.t. the square commutes, then
$$\begin{tikzcd} W \arrow[r, "t'"] \arrow[d, "s'"', "\sim"] & Y' \arrow[d, "t", "\sim"'] \\ X' \arrow[r, "f"'] & Y \end{tikzcd}$$
we obtain the comm. diagram $\rightsquigarrow$
$$\begin{tikzcd} & & W \arrow[dl, "s'"', "\sim"] \arrow[dr, "t'"] & & \\ & X' \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & Y' \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$
which yields the roof
$$\begin{tikzcd} & W \arrow[dl, "s \circ s'"', "\sim"] \arrow[dr, "g \circ t'"] & \\ X & & Z \end{tikzcd}$$
We denote this roof by $\frac{g}{t} \circ \frac{f}{s}$

Obs: It turns out such pair of arrows always exists. (Thus any two morphisms $X \longrightarrow Y$ and $Y \longrightarrow Z$ in $\mathcal{C}_S$ can be composed.)

This follows from the properties of the multiplicative system S. (Specifically, by (S3)).

Recall:

(S1) $1_X \in S$ for all $X \in \mathcal{C}$

(S2) If $f, g \in S$ are composable, then $g \circ f \in S$.

(S3) Any diagram of the form
$$\begin{tikzcd} & Z \arrow[d, "s \in S"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
(with $s \in S$) can be completed to a commutative sq.
$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"', "\sim"] & Z \arrow[d, "s", "\sim"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
with $t \in S$.

(S3') Same statement with the arrows reversed.

(S4) Let $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The foll. are eq:

(a) $\exists\, t \in S$: $t \circ f = t \circ g$

(b) $\exists\, s \in S$: $f \circ s = g \circ s$

Proposition. $\circ$ is well-defined. In other words,

If $\left[\frac{f}{s}\right] = \left[\frac{f'}{s'}\right]$ and $\left[\frac{g}{t}\right] = \left[\frac{g'}{t'}\right]$, then $\left[\frac{g}{t} \circ \frac{f}{s}\right] = \left[\frac{g'}{t'} \circ \frac{f'}{s'}\right]$.

(whenever the roofs are composable)

Proof. Suppose

$$
\begin{tikzcd}
 & A \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}
\sim
\begin{tikzcd}
 & A' \arrow[dl, "s'"', "\sim"] \arrow[dr, "f'"] & \\
X & & Y
\end{tikzcd}
\qquad \text{i.e. } \frac{f}{s} \sim \frac{f'}{s'}
$$

and

$$
\begin{tikzcd}
 & B \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\
Y & & Z
\end{tikzcd}
\sim
\begin{tikzcd}
 & B' \arrow[dl, "t'"', "\sim"] \arrow[dr, "g'"] & \\
Y & & Z
\end{tikzcd}
\qquad \text{i.e. } \frac{g}{t} \sim \frac{g'}{t'}
$$

We will show $\frac{g}{t} \circ \frac{f}{s} \sim \frac{g}{t} \circ \frac{f'}{s'}$ and $\frac{g}{t} \circ \frac{f'}{s'} \sim \frac{g'}{t'} \circ \frac{f'}{s'}$

whence the result follows by transitivity of $\sim$.

Since $\frac{f}{s} \sim \frac{f'}{s'}$, there exist morphisms $A'' \xrightarrow{a} A$, $A'' \xrightarrow{a'} A'$ and $A'' \xrightarrow{\alpha} X$ (α in S.)

s.t. the following diagram commutes

$$
\begin{tikzcd}
 & X & \\
A \arrow[ur, "s"] \arrow[dr, "f"'] & A'' \arrow[l, "a"'] \arrow[r, "a'"] \arrow[u, "\sim\alpha"] & A' \arrow[ul, "s'"'] \arrow[dl, "f'"] \\
 & Y &
\end{tikzcd}
$$

Now, by S3, we find arrows $C \xrightarrow{c_1} A$, $C \xrightarrow{c_2} B$

s.t. the foll. diag. commutes

$$
\begin{tikzcd}
 & & C \arrow[dl, "c_1"', "\sim"] \arrow[dr, "c_2"] & & \\
 & A \arrow[dl, "s"', "\sim"] \arrow[dr, "f"] & & B \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}
$$

By S3 again, we obtain the c.d.

$$
\begin{tikzcd}
 & & C' \arrow[dl, "c_1'"', "\sim"] \arrow[dr, "c_2'"] & & \\
 & A' \arrow[dl, "s'"', "\sim"] \arrow[dr, "f'"] & & B \arrow[dl, "t"', "\sim"] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}
$$

* Thus, we have obtained two roofs that we wish to show are equivalent:

i.e.

$$
\begin{tikzcd}
 & C \arrow[dl, "s \circ c_1"', "\sim"] \arrow[dr, "g \circ c_2"] & \\
X & & Z
\end{tikzcd}
\sim
\begin{tikzcd}
 & C' \arrow[dl, "s' \circ c_1'"', "\sim"] \arrow[dr, "g \circ c_2'"] & \\
X & & Z
\end{tikzcd}
$$

By consecutive application of (S3) the diagram

$$\begin{tikzcd}
& & C' \arrow[d, "\sim"', "c'_1"] \\
& A'' \arrow[r, "a'"] \arrow[d, "a"'] & A' \\
C \arrow[r, "\sim", "c_1"'] & A &
\end{tikzcd}$$

can be completed to ...

$$\begin{tikzcd}
E \arrow[r, "e'"] \arrow[d, "\sim"', "e"] & D' \arrow[r, "\partial'"] \arrow[d, "\sim"', "d'"] & C' \arrow[d, "\sim"', "c'_1"] \\
D \arrow[r, "\partial", "\sim"'] \arrow[d, "d"] & A'' \arrow[r, "a'"] \arrow[d, "a"] & A' \\
C \arrow[r, "\sim", "c_1"'] & A &
\end{tikzcd}$$

which commutes.

Then, the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "e'"] \arrow[d, "\sim"', "e"] \arrow[dr, "\partial\circ e", "\sim"'] & D' \arrow[r, "\partial'"] \arrow[d, "\sim"', "d'"] & C' \arrow[r, "c'_2"] \arrow[d, "\sim"', "c'_1"] & B \arrow[r, "g"] \arrow[d, "\sim"', "t"] & Z \\
D \arrow[r, "\partial", "\sim"'] \arrow[d, "d"] & A'' \arrow[r, "a'"] \arrow[d, "a"] \arrow[dr, "\alpha", "\sim"'] & A' \arrow[r, "f'"] \arrow[d, "\sim"', "s'"] & Y \arrow[dd, "1_Y"] & \\
C \arrow[r, "c_1", "\sim"'] \arrow[d, "c_2"] & A \arrow[r, "s", "\sim"'] \arrow[d, "f"] & X & & \\
B \arrow[r, "t", "\sim"'] \arrow[d, "g"] & Y \arrow[rr, "1_Y"] & & Y & \\
Z & & & &
\end{tikzcd}$$

Thus:

$$t\circ c_2\circ d\circ e = t\circ c'_2\circ \partial'\circ e'$$

By (S4), there exists $F \xrightarrow{\tau} E$ in S s.t.

$$c_2\circ d\circ e\circ \tau = c'_2\circ \partial'\circ e'\circ \tau$$

Let $h = d\circ e\circ \tau$,

$h' = \partial'\circ e'\circ \tau$.

It follows that the next diagram commutes:

$$
\begin{tikzcd}
F \arrow[r, "\partial' \circ e' \circ \tau"] \arrow[d, "d \circ e \circ \tau"'] \arrow[dr, "\alpha \circ \partial \circ e \circ \tau", "\sim"'] & C' \arrow[r, "c_2'"] \arrow[d, "s' \circ c_1'"] & B \arrow[dd, "g"] \\
C \arrow[r, "s \circ c_1"'] \arrow[d, "c_2"'] & X & \\
B \arrow[rr, "g"] & & Z
\end{tikzcd}
$$

But this is nothing else that the diagram

$$
\begin{tikzcd}
 & X & \\
C \arrow[ur, "s \circ c_1", "\sim"'] \arrow[dr, "g \circ c_2"'] & F \arrow[l, "h"'] \arrow[r, "h'"] \arrow[u, "\gamma"', "\sim"] & C' \arrow[ul, "s' \circ c_1'"', "\sim"] \arrow[dl, "g \circ c_2'"] \\
 & Z &
\end{tikzcd}
$$

Here:

$\gamma = \alpha \circ \partial \circ e \circ \tau \in S$

$h = d \circ e \circ \tau$

$h' = \partial' \circ e' \circ \tau$

Therefore

$$\frac{g \circ c_2}{s \circ c_1} \sim \frac{g \circ c_2'}{s' \circ c_1'} \quad \text{meaning} \quad \frac{g}{t} \circ \frac{f}{s} \sim \frac{g}{t} \circ \frac{f'}{s'}$$

Finally, we prove $\frac{g}{t} \circ \frac{f'}{s'} \sim \frac{g'}{t'} \circ \frac{f'}{s'}$.

We have the following c.d since we know $\frac{g}{t} \sim \frac{g}{t'}$,

$$(*) \quad \begin{tikzcd} & Y & \\ B \arrow[ur, "t", "\sim"'] \arrow[dr, "g"'] & P \arrow[u, "\pi"', "\sim"] \arrow[l, "p"'] \arrow[r, "p'"] & B' \arrow[ul, "t'"', "\sim"] \arrow[dl, "g'"] \\ & Z & \end{tikzcd}$$

with $\pi \in S$.

By (S3) the following diagram can be completed with the square at the top:

$$\begin{tikzcd} R \arrow[r, dashed, "r"] \arrow[d, dashed, "r'"', "\sim"] & P \arrow[r, "p"] \arrow[d, "\pi", "\sim"'] & B \arrow[r, "g"] & Z \\ A' \arrow[r, "f'"'] \arrow[d, "s'"', "\sim"] & Y & & \\ X & & & \end{tikzcd}$$

(with $r' \in S$)

which yields the roof

$$\begin{tikzcd} & R \arrow[dl, "s' \circ r'"', "\sim"] \arrow[dr, "g \circ p \circ r"] & \\ X & & Z \end{tikzcd}$$

Since $g \circ p = g' \circ p'$ by $(*)$, this roof is equal to

$$\begin{tikzcd} & R \arrow[dl, "s' \circ r'"'] \arrow[dr, "g' \circ p' \circ r"] & \\ X & & Z \end{tikzcd}$$

Therefore, these two roofs are equivalent.

The first roof belongs to $\left[\frac{g}{t} \circ \frac{f'}{s'}\right]$.

The second roof belongs to $\left[\frac{g'}{t'} \circ \frac{f'}{s'}\right]$.

Hence

$$\left[\frac{g'}{t'} \circ \frac{f'}{s'}\right] = \left[\frac{g}{t} \circ \frac{f'}{s'}\right]$$

The proof is complete. $\boxed{\text{Ꝝ}}$