

TRIANGULATED CATEGORIES AND REPRESENTATION THEORY

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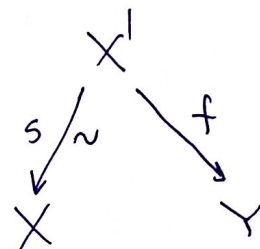
2024-09-25

Let $\mathcal{C}$ be a category and S a multiplicative system of $\mathcal{C}$. The localization of $\mathcal{C}$ by S , denoted $\mathcal{C}_S$, has as hom-sets

$$\text{Hom}_{\mathcal{C}_S}(X, Y) = \left\{ (X', s, f) \mid \begin{array}{l} X' \in \text{Ob } \mathcal{C}, s \in S \\ s \in \text{Hom}_{\mathcal{C}}(X', X), \\ f \in \text{Hom}_{\mathcal{C}}(X', Y) \end{array} \right\} / \sim$$

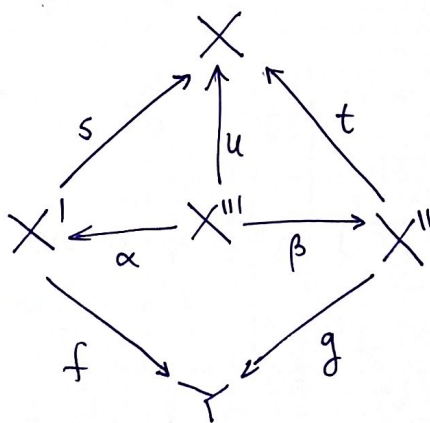
where $\sim$ is the equivalence relation defined below.

Notation. A triple (X', s, f) is drawn as the roof



and we denote it $\frac{f}{s}$. (The $\sim$ denotes $s \in S$)

Def. Given (X', s, f) and (X'', t, g) , we define $\frac{f}{s} \sim \frac{g}{t}$ iff there is $u \in S$, $\alpha, \beta \in \mathcal{C}$ s.t. the following diagram commutes



The equivalence class of the roof (X', s, f) is denoted $\left[\frac{f}{s} \right]$.

GOAL To show that composition of classes of roofs is well-defined.

In other words: $\frac{f}{s} \sim \frac{f'}{s'}$ and $\frac{g}{t} \sim \frac{g'}{t'} \Rightarrow \frac{g}{t} \circ \frac{f}{s} \sim \frac{g'}{t'} \circ \frac{f'}{s'}$
 for $\left[\frac{f}{s} \right] \in \text{Hom}_{\mathcal{C}_S}(X, Y)$ and $\left[\frac{g}{t} \right] \in \text{Hom}_{\mathcal{C}_S}(Y, Z)$.

Def of Composition Fix two roofs

If there exist a pair of arrows

$$X' \xleftarrow[\sim]{s'} W \xrightarrow{t'} Y'$$

s.t. the square commutes, then

$$\begin{array}{ccc} W & \xrightarrow{t'} & Y' \\ s' \downarrow \sim & & \sim \downarrow t \\ X' & \xrightarrow{f} & Y \end{array}$$

we obtain
the comm.
diagram
which
yields
the roof

$$\begin{array}{ccc} X' & & Y' \\ s \swarrow \sim & & \searrow f \\ X & & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} Y' & & Z \\ t \swarrow \sim & & \searrow g \\ Y & & Z \end{array}$$

$$\begin{array}{ccccc} & & W & & \\ & s' \swarrow \sim & & \searrow t' & \\ X' & & & & Y' \\ s \swarrow \sim & & f & & t \swarrow \sim \\ X & & Y & & Z \end{array}$$

$$\begin{array}{ccc} & W & \\ sos' \swarrow \sim & & \searrow got' \\ X & & Z \end{array}$$

We denote this roof
by $\frac{g}{t} \circ \frac{f}{s}$

Obs: It turns out such pair of arrows always exists.

(Thus any two morphisms $X \rightarrow Y$ and $Y \rightarrow Z$
in $\mathcal{C}_S$ can be composed.)

This follows from the properties of the multiplicative system S .
(Specifically, by (S3)).

Recall:

(S1) $1_X \in S$ for all $X \in \mathcal{C}$

(S2) If $f, g \in S$ are composable,
then $g \circ f \in S$.

(S3) Any diagram of the form
 $\begin{array}{ccc} & Z & \\ & \downarrow s \in S & \\ X & \xrightarrow{f} & Y \end{array}$
(with $s \in S$)
can be completed to a commutative sq.

(S3') Same statement with the
arrows reversed.

(S4) Let $f, g \in \text{Hom}_{\mathcal{C}}(X, Y)$.
The foll. are eq:

(a) $\exists t \in S : t \circ f = t \circ g$

(b) $\exists s \in S : f \circ s = g \circ s$

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ t \downarrow \sim & & \sim \downarrow s \\ X & \xrightarrow{f} & Y \end{array}$$

with $t \in S$.

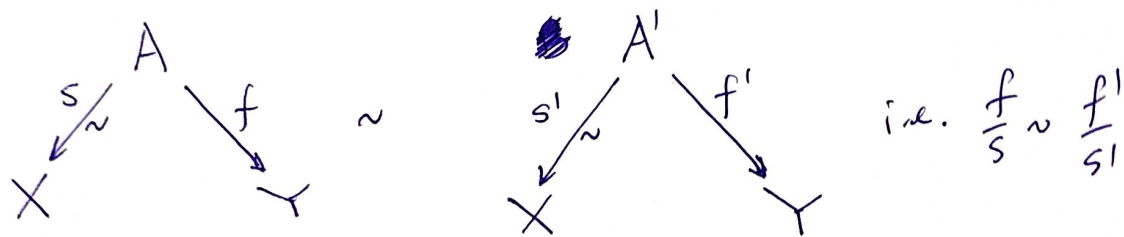
Proposition. $\circ$ is well-defined. In other words,

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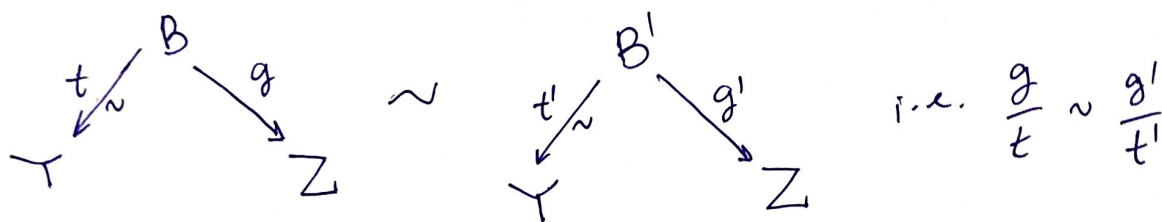
If $\begin{bmatrix} f \\ s \end{bmatrix} = \begin{bmatrix} f' \\ s' \end{bmatrix}$ and $\begin{bmatrix} g \\ t \end{bmatrix} = \begin{bmatrix} g' \\ t' \end{bmatrix}$, then $\begin{bmatrix} g \circ f \\ t \circ s \end{bmatrix} = \begin{bmatrix} g' \circ f' \\ t' \circ s' \end{bmatrix}$.
(whenever the roofs are composable)

Proof.

Suppose

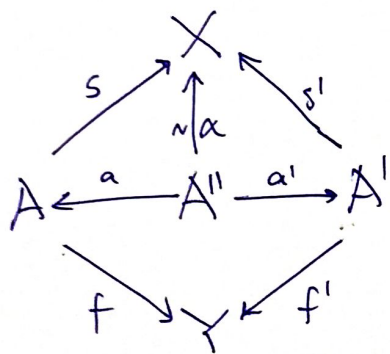


and

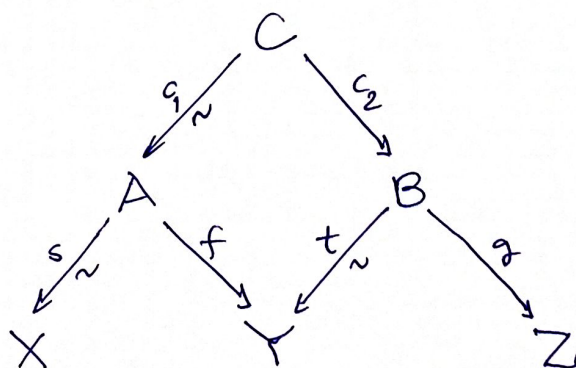


We will show $\frac{g}{t} \circ \frac{f}{s} \sim \frac{g}{t} \circ \frac{f'}{s'}$ and $\frac{g}{t} \circ \frac{f'}{s'} \sim \frac{g'}{t'} \circ \frac{f'}{s'}$
whence the result follows by transitivity of $\sim$.

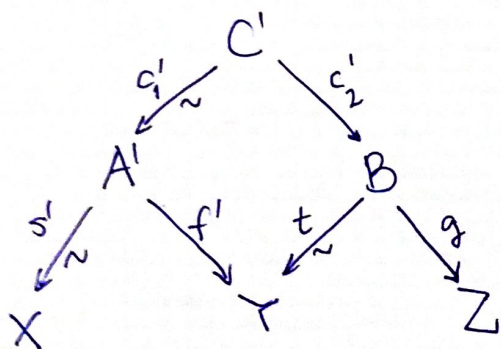
Since $\frac{f}{s} \sim \frac{f'}{s'}$, there exist morphisms $A'' \xrightarrow{a} A$ and $A'' \xrightarrow{a'} A'$ and $A'' \xrightarrow{\alpha} X$ s.t. the following diagram commutes
(α in S .)



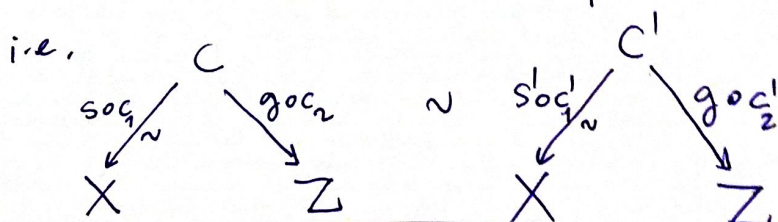
Now, by S3, we find arrows $C \xrightarrow{c_1} A$ and $C \xrightarrow{c_2} B$ s.t. the foll. diag. commutes



By S3 again, we obtain the c.d.

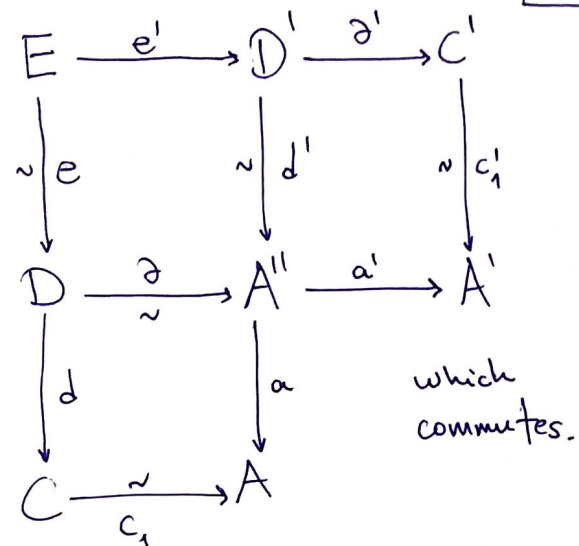
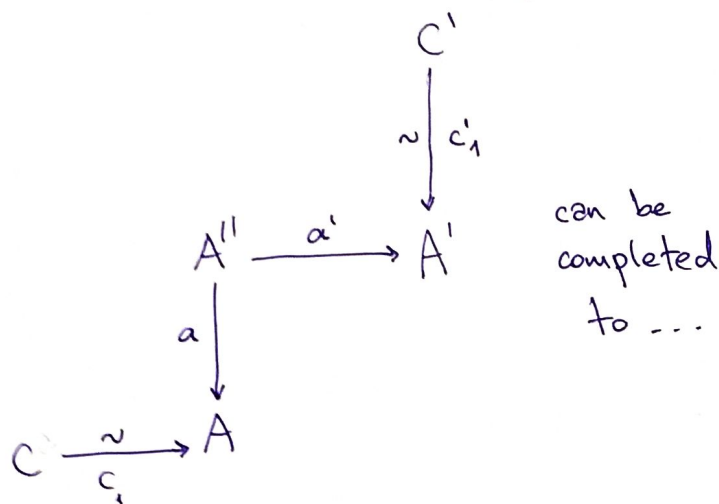


* Thus, we have obtained two roofs that we wish to show are equivalent:

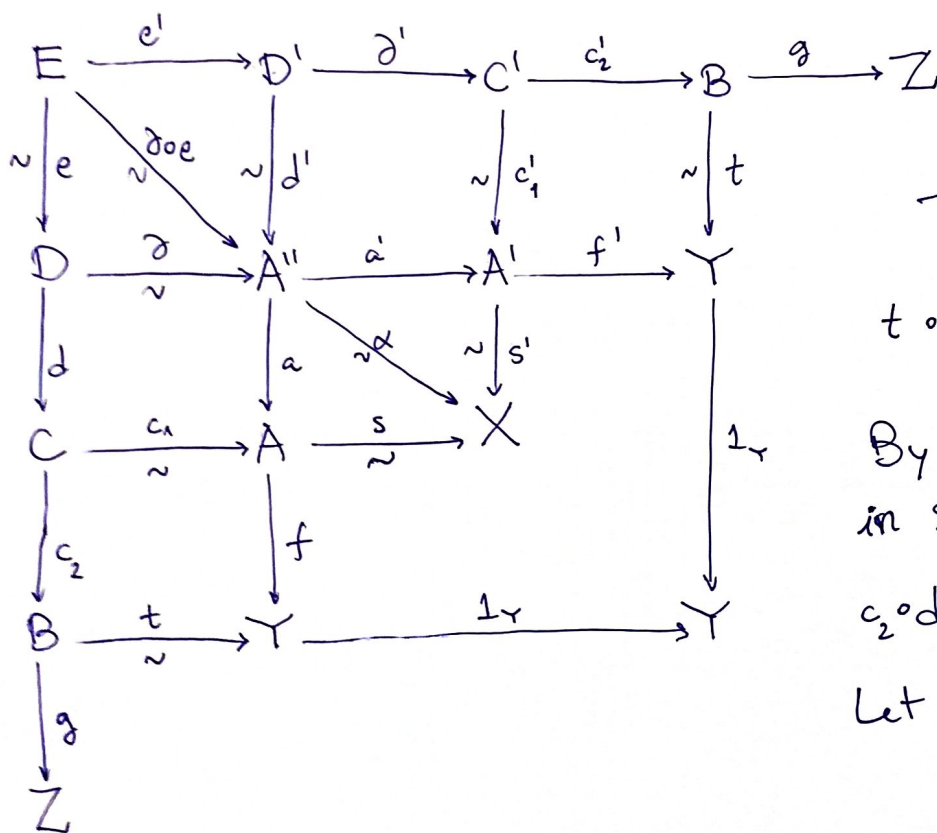


By consecutive application of (S3) the diagram

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Then, the following diagram commutes:



Thus:

$$t \circ c_2 \circ d \circ e = t \circ c'_2 \circ \partial' \circ e'$$

By (S4), there exists $F \xrightarrow{\tau} E$ in S s.t.

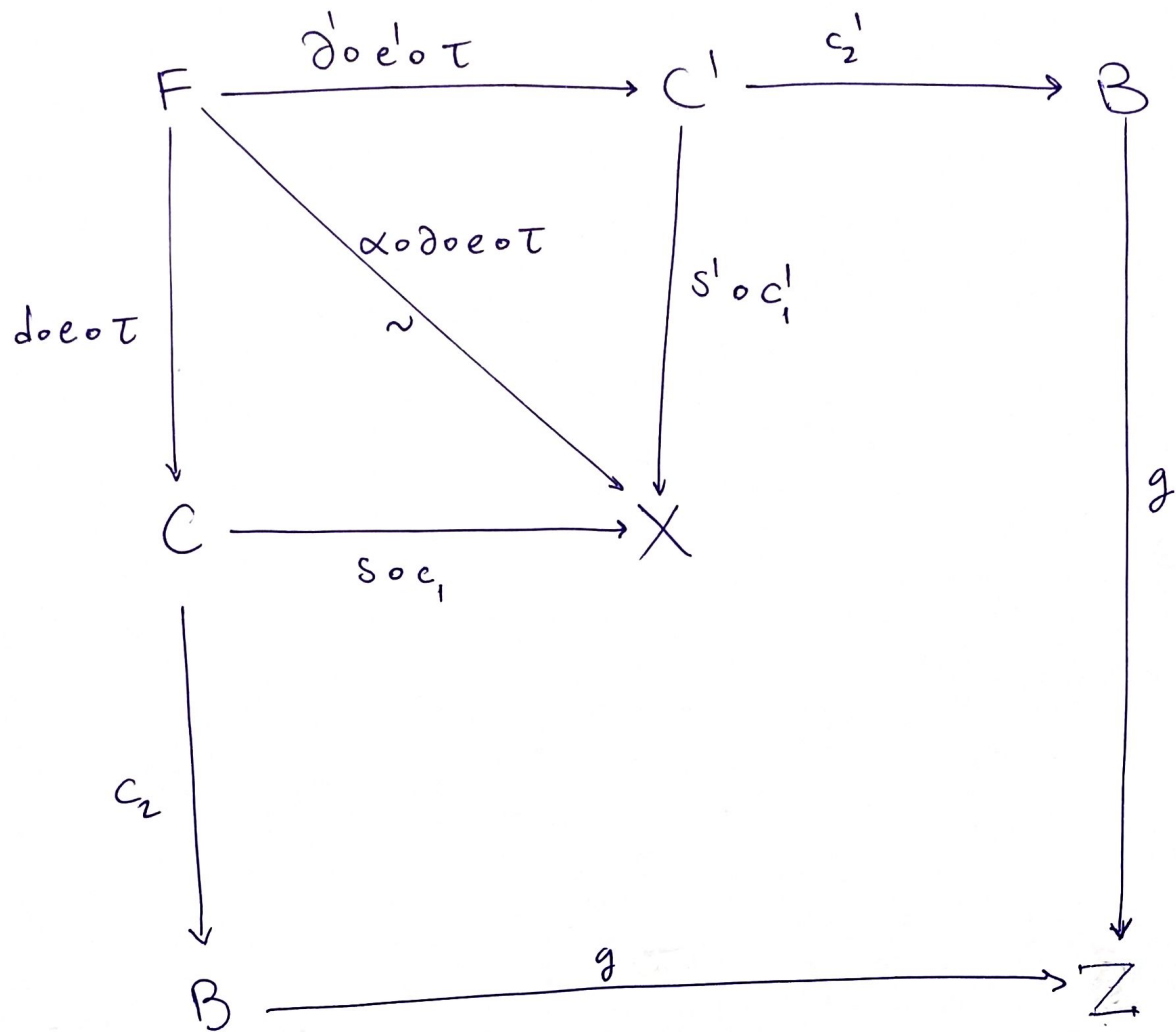
$$c_2 \circ d \circ e \circ \tau = c'_2 \circ \partial' \circ e' \circ \tau$$

$$\text{Let } h = d \circ e \circ \tau,$$

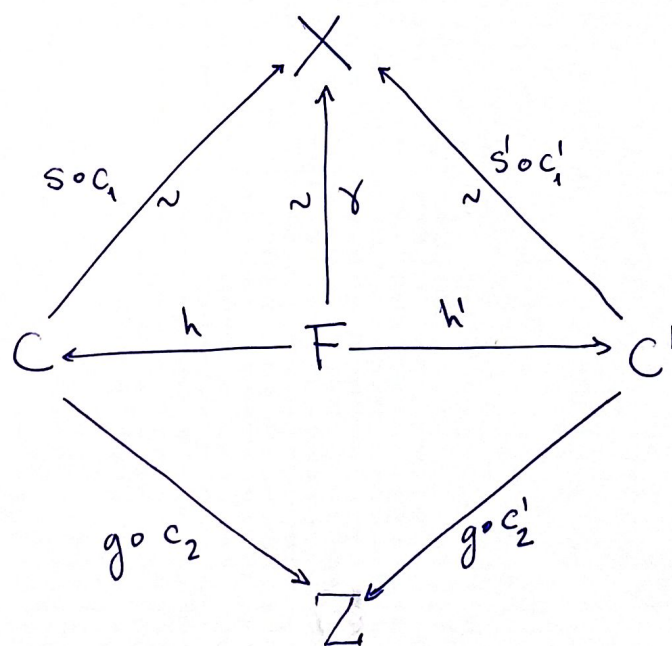
$$h' = \partial' \circ e' \circ \tau.$$

It follows that the next diagram commutes:

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But this is nothing else than the diagram



Here:

$$\gamma = \alpha \circ \delta \circ e \circ \tau \in S$$

$$h = doe \circ \tau$$

$$h' = \delta' \circ e' \circ \tau$$

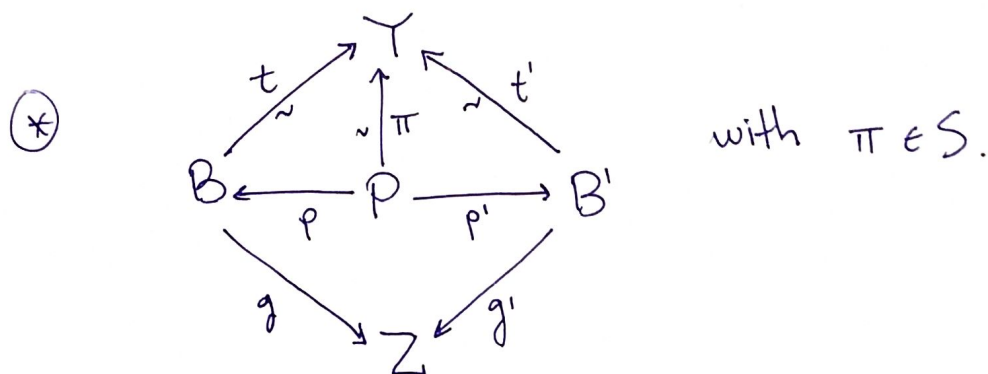
Therefore

$$\frac{g \circ c_2}{s \circ c_1} \sim \frac{g \circ c'_2}{s' \circ c'_1} \quad \text{meaning} \quad \frac{g}{t} \circ \frac{f}{s} \sim \frac{g}{t} \circ \frac{f'}{s'}$$

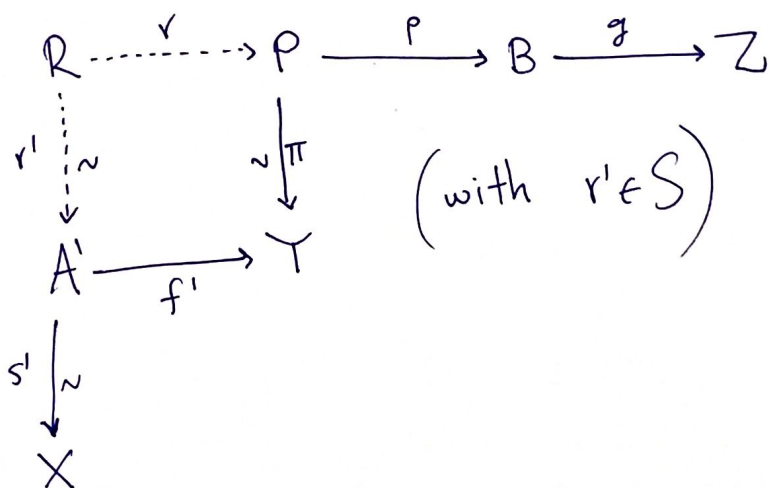
Finally, we prove $\frac{g}{t} \circ \frac{f'}{s'} \sim \frac{g'}{t'} \circ \frac{f'}{s'}$.

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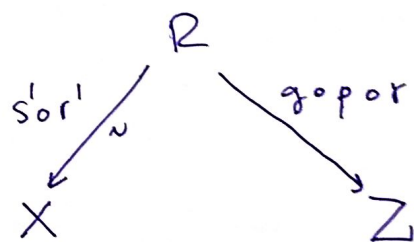
We have the following c.d. since we know $\frac{g}{t} \sim \frac{g}{t'}$,



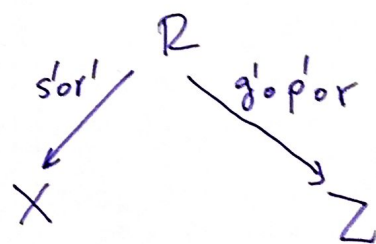
By (S3) the following diagram can be completed with the square at the top:



which yields the roof



Since $g \circ p = g' \circ p'$ by (*), this roof is equal to therefore, these two roofs are equivalent.



The first roof belongs to $\left[\frac{g}{t} \circ \frac{f'}{s'} \right]$.

The second roof belongs to $\left[\frac{g'}{t'} \circ \frac{f'}{s'} \right]$.

Hence

$$\left[\frac{g'}{t'} \circ \frac{f'}{s'} \right] = \left[\frac{g}{t} \circ \frac{f'}{s'} \right]$$

The proof is complete. $\square$