On the number of positive roots of the quadratic form of the quiver Dl,r

Let $\ell, r \in \mathbb{Z}_{\geq 1}$ and consider the bound quiver

$$\begin{tikzcd}
& & & a \arrow[dr] & & & \\
1 \arrow[r] & \cdots \arrow[r] & \ell \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \ell+1 \arrow[r] & \cdots \arrow[r] & \ell+r \\
& & & b \arrow[ur] & & &
\end{tikzcd}$$

denoted $D_{\ell,r}$.

The assoiated quadratic form is

$$q(\mathbf{x}) = x_a^2 + x_b^2 + \sum_{i=1}^{\ell} x_i^2 - \sum_{i=1}^{\ell-1} x_i x_{i+1}$$
$$+ \sum_{i=\ell+1}^{\ell+r} x_i^2 - \sum_{i=\ell+1}^{\ell+r-1} x_i x_{i+1}$$
$$- x_\ell x_a - x_\ell x_b - x_{\ell+1} x_a - x_{\ell+1} x_b + x_\ell x_{\ell+1}$$

where $\mathbf{x}$ is the vector $(x_1, \dots, x_{\ell+r}, x_a, x_b)$.

<u>GOAL</u> To show that the number of positive roots

$$\lambda = \#\{ \mathbf{x} \in \mathbb{Z}_{\geq 0}^{\ell+r+2} \mid q(\mathbf{x}) = 1 \}$$

is $(\ell+2)(\ell+1) + (r+2)(r+1) + \ell r - 2$.

<u>Obs</u> The number of nonnegative integer solutions of $2q = 2$ is the same as λ.

Known $2q$ can be written as a sum of squares as

$(*)$
$$2q=\sum_{i=1}^{\ell}(x_{i-1}-x_i)^2+\sum_{j=1}^{r}(x_{\ell+j}-x_{\ell+j+1})^2$$
$$+(x_\ell+x_{\ell+1}-x_a-x_b)^2+(x_a-x_b)^2$$

where $x_0=x_{\ell+r+1}=0$.

This is the same as the one of the note of 2024-07-24 but reindexed.

$$2q=\sum_{i=1}^{\ell-1}(x_i-x_{i+1})^2+x_1^2$$
$$+\sum_{i=\ell+1}^{\ell+r-1}(x_i-x_{i+1})^2+x_{\ell+r}^2$$
$$+(x_\ell+x_{\ell+r}-x_a-x_b)^2+(x_a-x_b)^2$$

Obs $q(x)=1$ iff exactly two squares in $(*)$ equal 1 and the others are zero (since we are looking for solutions in $\mathbb{Z}$)

We consider several cases:

(I) suppose that $x_{\ell+1}=\cdots=x_{\ell+r}=0$ (the right arm is zero)

Then
$$2q=\sum_{i=1}^{\ell}(x_{i-1}-x_i)^2+(x_\ell-x_a-x_b)^2+(x_a-x_b)^2$$
and the $\underbrace{\text{two nonzero summands}}_{\text{TNS}}$ are somewhere in this expression.

(I') Suppose the opposite of (I). We have two cases: either $x_1=\cdots=x_\ell=0$ or not. In the first case, the TNS appear in the following simplified polynomial:

$$2q = \sum_{j=1}^{r} (x_{\ell+j} - x_{\ell+j+1})^2 + (x_\ell + x_{\ell+1} - x_a - x_b)^2 + (x_a - x_b)^2$$

In the second case, neither of the sums

$$\sum_{i=1}^{\ell} (x_{i-1} - x_i)^2 \quad , \quad \sum_{j=1}^{r} (x_{\ell+j} - x_{\ell+j+1})^2$$

can be zero because at least one of $x_1, \dots, x_\ell$ is nonzero and at least one of $x_{\ell+1}, \dots, x_{\ell+r}$ is nonzero.
This forces

$$(x_\ell + x_{\ell+1} - x_a - x_b)^2 = (x_a - x_b)^2 = 0.$$

Therefore, $*$ simplifies to

$$2q = \sum_{i=1}^{\ell} (x_{i-1} - x_i)^2 + \sum_{j=1}^{r} (x_{\ell+j} - x_{\ell+j+1})^2$$

Therefore, in total we have three cases.

(I) Right arm zero

(II) Left arm zero and right arm nonzero.

(III) Both arms nonzero

$\boxed{\text{CASE (I)}}$ The TNS appear among the following terms:

$$2q = \sum_{i=1}^{\ell} (x_{i-1} - x_i)^2 + (x_\ell - x_a - x_b)^2 + (x_a - x_b)^2$$

Either $x_1 = \cdots = x_\ell = 0$ or not. In the first case,

$\boxed{\text{CASE I.1}}$

$$\sum_{i=1}^{\ell} (x_{i-1} - x_i)^2 = 0, \quad (\underbrace{x_\ell}_{0} - x_a - x_b)^2 = 1, \quad (x_a - x_b)^2 = 1$$

so $|x_a + x_b| = 1$ and $|x_a - x_b| = 1$. The only possibilities are

$$x_a = 1 \text{ and } x_b = 0$$ which corresponds to the indecomposable

$$\begin{tikzcd}[row sep=small, column sep=small] & & & 1 \arrow[dr] & & & \\ 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[r] & \cdots \arrow[r] & 0 \\ & & & 0 \arrow[ur] & & & \end{tikzcd}$$

or

$$x_a = 0 \text{ and } x_b = 1$$ which corresponds to the indecomposable

$$\begin{tikzcd}[row sep=small, column sep=small] & & & 0 \arrow[dr] & & & \\ 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[r] & \cdots \arrow[r] & 0 \\ & & & 1 \arrow[ur] & & & \end{tikzcd}$$

COUNT of positive roots = 2.

Suppose now that it is not true that $x_1 = \cdots = x_\ell = 0$. Then the sum $\sum_{i=1}^{\ell} (x_{i-1} - x_i)^2$ contains at least one nonzero term, say $(x_{j-1} - x_j)^2 = 1$. The other terms have to be 0 so

with $1 \leq j \leq \ell$

$$(x_0 - x_1)^2 = \cdots = (x_{j-2} - x_{j-1})^2 = 0,$$ the first index s.t this is true.

whence

$$0 = x_0 = x_1 = \cdots = x_{j-1}.$$

We get $x_j = 1$.

Let's consider the possible options for the other nonzero term.

CASE I.2 Sup. that it appears also in $\sum_{i=1}^{\ell}(x_{i-1}-x_i)^2$, say $(x_{k-1}-x_k)^2 = 1$ with $k>j$ (WLOG).
Then all other terms are zero, so

- $(x_j - x_{j+1})^2 = \cdots = (x_{k-2} - x_{k-1})^2 = 0$
  whence $1 = x_j = x_{j+1} = \cdots = x_{k-1}$

- $(x_k - x_{k+1})^2 = \cdots = (x_{\ell-1} - x_\ell)^2 = 0$
  whence $x_k = \cdots = x_\ell$

- $(x_\ell - x_a - x_b)^2 = (x_a - x_b)^2 = 0$
  whence $x_a = x_b$ and $x_\ell = 2x_a$.

Since $x_{k-1} = 1$, we get $(1-x_k)^2 = 1$, i.e. $|1-x_k| = 1$
There are two possibilities:
either $x_k = 0$, so the second point above gives $x_k = \cdots = x_\ell = 0$ and from the third $x_a = x_b = 0$. Thus we get the indecomposable

$$\begin{tikzcd}[column sep=small] & & & & & & & & & 0 \arrow[dr] & & & \\ 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \underbrace{1 \arrow[r] & \cdots \arrow[r] & 1}_{j,\dots,k-1} \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[r] & \cdots \arrow[r] & 0 \\ & & & & & & & & & 0 \arrow[ur] & & & \end{tikzcd}$$

or $x_k = 2$, so $x_k = \cdots = x_\ell = 2$ and $x_a = x_b = 1$, which gives the indecomposable

$$\begin{tikzcd}[column sep=small] & & & & & & & & & 1 \arrow[dr] & & & \\ 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \underbrace{1 \arrow[r] & \cdots \arrow[r] & 1}_{j,\dots,k-1} \arrow[r] & \underbrace{2 \arrow[r] & \cdots \arrow[r] & 2}_{k,\dots,\ell} \arrow[ur] \arrow[dr] & & 0 \arrow[r] & \cdots \arrow[r] & 0 \\ & & & & & & & & & 1 \arrow[ur] & & & \end{tikzcd}$$

Since $1 \leq j < k \leq \ell$, there are exactly two options for j and k out of ℓ, giving $\binom{\ell}{2}$ positive roots. Since $x_k = 0$ or $x_k = 2$, in total for this case we get

$$2\binom{\ell}{2} = \ell(\ell-1)$$

positive roots.

$$\text{COUNT} = 2 + \ell(\ell-1)$$

<u>CASE I.3</u> Sup. the other nonzero term is $(x_\ell - x_a - x_b)^2$. From the current hypothesis, we get

- $(x_j - x_{j+1})^2 = \cdots = (x_{\ell-1} - x_\ell)^2 = 0$
  whence $1 = x_j = x_{j+1} = \cdots = x_\ell$

- $(x_a - x_b)^2 = 0$ whence $x_a = x_b$

Since $(x_\ell - x_a - x_b)^2 = 1$ and $x_\ell = 1$, we get $|1 - 2x_a| = 1$. which forces either $x_a = x_b = 0$ giving the indecomposable

$$\begin{tikzcd}[row sep=small, column sep=small]
 & & & & & & & & & 0 \arrow[dr] & & & \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & 1 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & 1 \arrow[ur] \arrow[dr] & & 0 \arrow[r] & \cdots \arrow[r] & 0 \\
 & & & j & \cdots & & & & \ell & 0 \arrow[ur] & & &
\end{tikzcd}$$

or $x_a = x_b = 1$ which gives

$$\begin{tikzcd}[row sep=small, column sep=small]
 & & & & & & & & & 1 \arrow[dr] & & & \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & 1 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & 1 \arrow[ur] \arrow[dr] & & 0 \arrow[r] & \cdots \arrow[r] & 0 \\
 & & & j & \cdots & & & & \ell & 1 \arrow[ur] & & &
\end{tikzcd}$$

For both options, j can take ℓ values, so we get 2ℓ positive roots

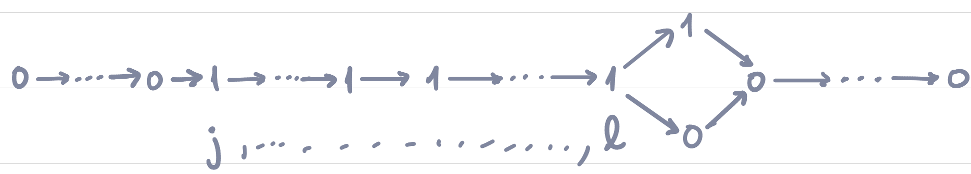
COUNT $= 2 + \ell(\ell-1) + 2\ell$

CASE I.4 Sup. the other nonzero term is $(x_a - x_b)^2$.
From the current hypothesis, we get

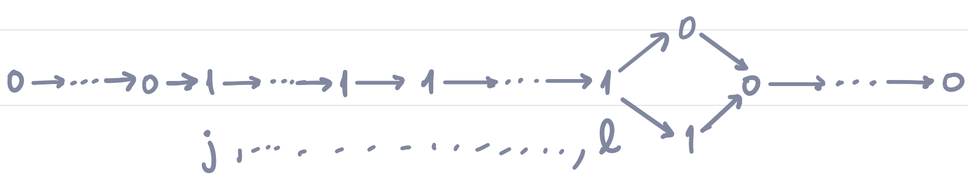
- $(x_j - x_{j+1})^2 = \cdots = (x_{\ell-1} - x_\ell)^2 = 0$
  whence $1 = x_j = x_{j+1} = \cdots = x_\ell$

- $(x_\ell - x_a - x_b)^2 = 0$ whence $x_\ell = x_a + x_b$

Since $x_\ell = 1$, we get $x_a + x_b = 1$. And from the assumption, $|x_a - x_b| = 1$, so either $(x_a, x_b) = (1,0)$ which gives



or $(x_a, x_b) = (0,1)$ which gives



Again, for both situations j can take ℓ values, so we get 2ℓ positive roots. COUNT $= 2 + \ell(\ell-1) + 2\ell + 2\ell$

So far, the cont is $2 + \ell^2 - \ell + 4\ell = \ell^2 + 3\ell + 2$
$= (\ell+2)(\ell+1)$

$\boxed{\text{CASE II}}$ Left arm zero and right arm nonzero.

This case is similar to (I), and using an analogous argument we get

$$2 + r(r-1) + 2r + 2r = (r+2)(r+1).$$

Since the representations

$$\begin{tikzcd}[column sep=small, row sep=tiny] & & & 1 \arrow[dr] & & & \\ 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[r] & \cdots \arrow[r] & 0 \\ & & & 0 \arrow[ur] & & & \end{tikzcd}, \quad \begin{tikzcd}[column sep=small, row sep=tiny] & & & 0 \arrow[dr] & & & \\ 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[r] & \cdots \arrow[r] & 0 \\ & & & 1 \arrow[ur] & & & \end{tikzcd}$$

get conted in both (I) and (II), we substract 2 from the total cont obtaining the following so far

$$\text{COUNT} = (\ell+2)(\ell+1) + (r+2)(r+1) - 2.$$

$\boxed{\text{CASE III}}$ Both arms nonzero

By the discussion on ☺, we have

$$2q = \sum_{i=1}^{l} (x_{i-1} - x_i)^2 + \sum_{j=1}^{r} (x_{l+j} - x_{l+j+1})^2 ,$$

$$\text{and} \quad (x_l + x_{l+1} - x_a - x_b)^2 = (x_a - x_b)^2 = 0$$

In order to satisfy $2q = 2$, there must be $1 \leq i \leq l$ and $1 \leq j \leq r$ s.t.

$$(x_{i-1} - x_i)^2 = 1 \quad \text{and} \quad (x_{l+j} - x_{l+j+1})^2 = 1$$

All other terms are zero, so

$$0 = x_0 = x_1 = \cdots = x_{i-2} = x_{i-1} ,$$
$$x_i = x_{i+1} = \cdots = x_l$$

and

$$x_{l+1} = x_{l+2} = \cdots = x_{l+j-1} = x_{l+j} ,$$
$$x_{l+j+1} = x_{l+j+2} = \cdots = x_{l+r} = x_{l+r+1} = 0 .$$

Since $x_{i-1} = 0$, we get $x_i = (0 - x_i)^2 = 1$, whence the chain above becomes $x_i = x_{i+1} = \cdots = x_l = 1$.

Since $x_{l+j+1} = 0$, we get $x_{l+j} = (x_{l+j} - 0)^2 = 1$, whence the chain above becomes $x_{l+1} = x_{l+2} = \cdots = x_{l+j-1} = x_{l+j} = 1$

Moreover, since $(x_\ell + x_{\ell+1} - x_a - x_b)^2 = (x_a - x_b)^2 = 0$ we get

$$\underset{\substack{\shortparallel \\ 1}}{x_\ell} + \underset{\substack{\shortparallel \\ 1}}{x_{\ell+1}} = 2x_a, \qquad x_a = x_b$$

Thus, $2 = 2x_a$ gives $x_a = x_b = 1$.
We have obtained the positive root

$$\begin{array}{cccccccccccccccccc}
(0, & \ldots, & 0, & 1, & \ldots, & 1, & 1 & , \ldots, & 1, & 0, & \ldots, & 0 & , 1, & 1) \\
1, & \ldots, & i-1, & i, & \ldots, & \ell, & \ell+1, & \ldots, & \ell+j, & \ldots & \ldots, & \ell+r, & a, & b
\end{array}$$

which corresponds to the indecomposable

$$\begin{tikzcd}
 & & & & & & 1 \arrow[dr] & & & & & & \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \underset{i}{1} \arrow[r] & \cdots \arrow[r] & \underset{\ell}{1} \arrow[ur] \arrow[dr] & & \underset{\ell+1}{1} \arrow[r] & \cdots \arrow[r] & \underset{\ell+j}{1} \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \\
 & & & & & & 1 \arrow[ur] & & & & & &
\end{tikzcd}$$

Note that i can take ℓ values, while j can take r values. This means we get a total of ℓr positive roots for this case. Our total count among cases (I), (II) and (III) is

$$\text{COUNT} = (\ell+2)(\ell+1) + (r+2)(r+1) - 2 + \ell r$$

which is the one of our goal.
The solution is complete.