TRIANGULATED CATEGORIES AND REPRESENTATION THEORY # 2024-10-10

* Define the relation $\sim$ over $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A^\bullet, B^\bullet)$ by

$$f^\bullet \sim g^\bullet \iff \forall i \in \mathbb{Z}, \exists h^i \in \mathrm{Hom}_{\mathcal{A}}(A^i, B^{i-1}) \text{ s.t.}$$

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$$

Here: $\mathcal{A}$ is an abelian category and $\mathcal{C}(\mathcal{A})$ is the category of cochain complexes over $\mathcal{A}$.

<u>Homework</u>: $\sim$ is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A^\bullet, B^\bullet)$.

<u>Proof.</u> (I) We have $f^\bullet \sim f^\bullet$ for every $f^\bullet \in \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A^\bullet, B^\bullet)$ since taking $h^i = 0$ for all $i \in \mathbb{Z}$ gives

$$0 = f^i - f^i = d_B^{i-1} \circ h^i + h^{i+1} \circ d_A^i = 0$$

(II) Suppose $f^\bullet \sim g^\bullet$ and let $(h^i)_{i\in\mathbb{Z}}$ be an homotopy from $f^\bullet$ to $g^\bullet$. Define $\hat{h}^i = -h^i$ for all $i \in \mathbb{Z}$. Then, for all $i \in \mathbb{Z}$:

$$g^i - f^i = -(f^i + g^i) = -\left(h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i\right)$$

$$= (-h^{i+1}) \circ d_A^i + d_B^{i-1} \circ (-h^i) = \hat{h}^{i+1} \circ d_A^i + d_B^{i-1} \circ \hat{h}^i$$

Thus $g^\bullet \sim f^\bullet$.

(III) Let $F : f^\bullet \sim g^\bullet$ and $G : g^\bullet \sim h^\bullet$. We have

$$f^i - g^i = d_B^{i-1} \circ F^i + F^{i+1} \circ d_A^i$$

$$g^i - h^i = d_B^{i-1} \circ G^i + G^{i+1} \circ d_A^i \qquad , \quad i \in \mathbb{Z}$$

Thus,

$$f^i - h^i = d_B^{i-1} \circ (F^i + G^i) + (F^{i+1} + G^{i+1}) \circ d_A^i \quad , \quad i \in \mathbb{Z}$$

whence $f^\bullet \sim h^\bullet$.