Homology Functor on an Abelian Category

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Abstract

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Let $\mathcal{A}$ be an abelian category.

A cochain complex in $\mathcal{A}$ is a family $(A^n, d^n)_{n\in\mathbb{Z}}$ where A^n is an object in $\mathcal{A}$ and $d^n\colon A^n \to A^{n+1}$ is a morphism in $\mathcal{A}$ such that $d^{n+1} \circ d^n = 0$, for all $n \in \mathbb{Z}$. For brevity, we write $A^\bullet = (A^n, d^n)_{n\in\mathbb{Z}}$.

$$\cdots \to A^n \xrightarrow{d^n} A^{n+1} \xrightarrow{d^{n+1}} A^{n+2} \to \cdots$$

A morphism of cochain complexes, say from $A^\bullet = (A^n, d^n_A)_{n\in\mathbb{Z}}$ to $B^\bullet = (B^n, d^n_B)_{n\in\mathbb{Z}}$, is a family $(f^n)_{n\in\mathbb{Z}}$ of morphisms $f^n\colon A^n \to B^n$ such that $f^{n+1} \circ d^n_A = d^n_B \circ f^n$ for all $n \in \mathbb{Z}$, i.e., such that the following diagram commutes:

$$\begin{tikzcd}
A^n \arrow[r, "d^n_A"] \arrow[d, "f^n"] & A^{n+1} \arrow[d, "f^{n+1}"] \\
B^n \arrow[r, "d^n_B"] & B^{n+1}
\end{tikzcd}$$

We denote $f^\bullet = (f^n)_{n\in\mathbb{Z}}$. Two morphisms of cochain complex can be composed componentwise: $f^\bullet \circ g^\bullet = (f^n \circ g^n)_{n\in\mathbb{Z}}$. Thus, we can form the category of cochain complexes in $\mathcal{A}$, denoted $\mathrm{CoCh}(\mathcal{A})$. Furtheremore, it can be proved that $\mathrm{CoCh}(\mathcal{A})$ is abelian.

1 Kernels and cokernels

Just recall the definitions ans fix the notations.

In addition we present the following useful lemma.

Let $f \in \hom_{\mathcal{A}}(X, Y)$. There is a unique epimorphism $\overline{f}\colon X \to \operatorname{Im} f$ such that $f = \operatorname{im} f \circ \overline{f}$, i.e., such that the diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "\overline{f}"'] & & Y \\ & \operatorname{Im} f \arrow[ur, "\operatorname{im} f"'] & \end{tikzcd}$$

Proof. Ruiter. □

2 Cohomology functors

If $\mathcal{A} = \mathbf{Ab}$, the category of abelian groups, the nth homology group of a chain complex $A^\bullet$ is the quotient group $H^n(A^\bullet) = \operatorname{Ker} d_A^n / \operatorname{Im} d_A^{n-1}$.

Moreover, given a morphism of cochain complexes $f^\bullet\colon A^\bullet \to B^\bullet$, there is a induced morphism of abelian groups $H^n(f^\bullet)\colon H^n(A^\bullet) \to H^n(B^\bullet)$ defined by

$$a + \operatorname{Im} d_A^{n-1} \mapsto f^n(a) + \operatorname{Im} d_B^{n-1}$$

where $a \in \operatorname{Ker} d_A^n$.

Our objective is to generalize this constructions to an arbitrary abelian category.

2.1 Homology functor in objects

Recall that if $\psi\colon A \to B$ is a morphism of Abelian groups, its cokernel is the group $\operatorname{Coker} \psi = B / \operatorname{Im} \psi$. If in addition ψ is injective, we have $A \cong \operatorname{Im} \psi$, and hence $\operatorname{Coker} \psi \cong B/A$. In other words, the cokernel of an injective morphism of abelian groups is nothing else than the quotient of its codomain by its domain.

Thus if we want to define the analog of the quotient $\operatorname{Ker} d_A^n / \operatorname{Im} d_A^{n-1}$ in an abelian category, we must construct a canonical morphism $\theta_A^n\colon \operatorname{Im} d_A^{n-1} \to \operatorname{Ker} d_A^n$ and then define $H^n(A^\bullet)$ to be the cokernel of θ_A^n.

Recall that if f is a morphism in $\mathcal{A}$, its image is defined to be Ker(coker f). Likewise, its coimage is Coker(ker f).

In order to define θ_A^n we use the universal property of $\operatorname{Ker} d_A^n$, since it is an attracting object. First, let us see $d_A^n \circ \operatorname{im} d_A^{n-1} = 0$. We know $d_A^{n-1} = \operatorname{im} d_A^{n-1} \circ \overline{d_A^{n-1}}$, so

$$0 = d_A^n \circ d_A^{n-1} = d_A^n \circ \operatorname{im} d_A^{n-1} \circ \overline{d_A^{n-1}}$$

and since $\overline{d_A^{n-1}}$ is an epimorphism, $d_A^n \circ \operatorname{im} d_A^{n-1} = 0$. The universal property of the kernel of d_A^n implies there is a unique morphism $\operatorname{Im} d_A^{n-1} \to \operatorname{Ker} d_A^n$, which we denote θ_A^n, such that the following diagram commutes

$$
\begin{tikzcd}
\ker d_A^n \arrow[r, "\ker d_A^n"] & A^n \arrow[r, "d_A^n"] & A^{n+1} \\
\operatorname{Im} d_A^{n-1} \arrow[u, "\theta_A^n"] \arrow[ur, "\operatorname{im} d_A^{n-1}"'] & &
\end{tikzcd}
$$

that is,

$$\operatorname{im} d_A^{n-1} = \ker d_A^n \circ \theta_A^n. \tag{1}$$

Note there is no other possible map from the image to the kernel, so it is canonical. there is no other possible way to define such a map.

Let us see θ_A^n is monic.

The nth homology object of $A^\bullet$ is

$$H^n(A^\bullet) = \operatorname{Coker} \theta_A^n.$$

2.2 Homology functor in morphisms

Let $f^\bullet \colon A^\bullet \to B^\bullet$ be a morphism of cochain complexes. Our objective is to construct a canonical morphism $H^n(A^\bullet) \to H^n(B^\bullet)$, that is, from $\operatorname{Coker} \theta_A^n$ to $\operatorname{Coker} \theta_B^n$. The strategy is the same: we use the universal property of the cokernel of θ_A^n, since this is a repelling object.

We will construct morphisms α and β such that the following square commutes.

$$
\begin{tikzcd}
\operatorname{Im} d_A^{n-1} \arrow[r, "\theta_A^n"] \arrow[d, "\alpha"] & \operatorname{Ker} d_A^n \arrow[d, "\beta"] \\
\operatorname{Im} d_A^{n-1} \arrow[r, "\theta_B^n"] & \operatorname{Ker} d_B^n
\end{tikzcd}
$$

2.2.1 Construction of α

Since $\operatorname{Im} d_A^{n-1}$ is an attracting object, it is enoguh to show that a suitable morphism from $\operatorname{Im} d_A^{n-1}$ to B^n composed with $\operatorname{coker} d_B^{n-1}$ gives zero. What is such a morphism? Let us illustrate the current situation with the following diagram

$$
\begin{tikzcd}
\operatorname{Im} d_A^{n-1} \arrow[d, "\theta_A^n"'] \arrow[dr, "\operatorname{im} d_A^{n-1}"] & A^{n-1} \arrow[l, "\overline{d_A^{n-1}}"'] \arrow[r, "f^{n-1}"] \arrow[d, "d_A^{n-1}"] & B^{n-1} \arrow[r, "\overline{d_B^{n-1}}"] \arrow[d, "d_B^{n-1}"'] & \operatorname{Im} d_B^{n-1} \arrow[d, "\theta_B^n"] \arrow[dl, "\operatorname{im} d_B^{n-1}"'] \\
\operatorname{Ker} d_A^n \arrow[r, "\ker d_A^n"'] & A^n \arrow[r, "f^n"'] & B^n & \operatorname{Ker} d_B^n \arrow[l, "\ker d_B^n"]
\end{tikzcd}
$$

We see that we can choose $f^n \circ \operatorname{im} d_A^{n-1}$. Observe that

$$\begin{aligned}
\operatorname{coker} d_B^{n-1} \circ f^n \circ \operatorname{im} d_A^{n-1} \circ \overline{d_A^{n-1}} &= \operatorname{coker} d_B^{n-1} \circ f^n \circ d_A^{n-1} \\
&= \operatorname{coker} d_B^{n-1} \circ d_B^{n-1} \circ f^{n-1} \\
&= 0 \circ f^{n-1} = 0
\end{aligned}$$

Since $\overline{d_A^{n-1}}$ is an epimorphism, we get $\operatorname{coker} d_B^{n-1} \circ f^n \circ \operatorname{im} d_A^{n-1} = 0$. Thus, by the universal property of the kernel of $\operatorname{Coker} d_B^{n-1}$, that is, of $\operatorname{Im} d_B^{n-1}$, there is a unique morphism $\alpha\colon \operatorname{Im} d_A^{n-1} \to \operatorname{Im} d_B^{n-1}$ such that

$$f^n \circ \operatorname{im} d_A^{n-1} = \operatorname{im} d_B^{n-1} \circ \alpha \tag{2}$$

i.e., making the following diagram commute

$$\begin{tikzcd}
\operatorname{Im} d_B^{n-1} \arrow[r, "\operatorname{im} d_B^{n-1}"] & B^n \arrow[r, "\operatorname{coker} d_B^{n-1}"] & \operatorname{Coker} d_B^{n-1} \\
\operatorname{Im} d_A^{n-1} \arrow[u, dashed, "\alpha"] \arrow[ur, "f^n \circ \operatorname{im} d_A^{n-1}"'] & &
\end{tikzcd}$$

2.2.2 Construction of β.

This case is easier and follows the same pattern. Note that

$$d_B^n \circ (f^n \circ \ker d_A^n) = (d_B^n \circ f^n) \circ \ker d_A^n = f^{n+1} \circ d_A^n \circ \ker d_A^n = f^{n+1} \circ 0 = 0.$$

Hence there is a unique morphism $\beta\colon \operatorname{Ker} d_A^n \to \operatorname{Ker} d_B^n$ such that

$$f^n \circ \ker d_A^n = \ker d_B^n \circ \beta \tag{3}$$

2.3 Commutativity of the square

In order to show that the following square commutes, it is enoguh to show that $\ker d_B^n \circ (\theta_B^n \circ \alpha) = \ker d_B^n \circ (\beta \circ \theta_A^n)$ because $\ker d_B^n$ is monic.

$$\begin{tikzcd}
\operatorname{Im} d_A^{n-1} \arrow[r, "\theta_A^n"] \arrow[d, "\alpha"] & \operatorname{Ker} d_A^n \arrow[d, "\beta"] \\
\operatorname{Im} d_A^{n-1} \arrow[r, "\theta_B^n"] & \operatorname{Ker} d_B^n
\end{tikzcd}$$

Note that

$$\begin{aligned}
\ker d_B^n \circ \beta \circ \theta_A^n &= f^n \circ \ker d_A^n \circ \theta_A^n && \text{by (3)} \\
&= f^n \circ \operatorname{im} d_A^{n-1} && \text{by (1)} \\
&= \operatorname{im} d_B^{n-1} \circ \alpha && \text{by (2)} \\
&= \ker d_B^n \circ \theta_B^n \circ \alpha && \text{by (3)}
\end{aligned}$$

and the claim follows.

2.4 Construction of $H^n(f^\bullet)$

So far, we know that the first square in the following diagram commutes. Our final objective is to demonstrate that there exists a morphism $\operatorname{Coker} \theta_A^n \to \operatorname{Coker} \theta_B^n$ that makes the second square commute as well. Since this morphism is unique, as we will show, the only possible choice is to define $H^n(f^\bullet)$ as this morphism.

$$\begin{tikzcd}
\operatorname{Im} d_A^{n-1} \arrow[r, "\theta_A^n"] \arrow[d, "\alpha"] & \operatorname{Ker} d_A^n \arrow[r, "\operatorname{coker} \theta_A^n"] \arrow[d, "\beta"] & \operatorname{Coker} \theta_A^n \arrow[d, "H^n(f^\bullet)"] \\
\operatorname{Im} d_A^{n-1} \arrow[r, "\theta_B^n"] & \operatorname{Ker} d_B^n \arrow[r, "\operatorname{coker} \theta_B^n"] & \operatorname{Coker} \theta_B^n
\end{tikzcd}$$

Since cokernels are repelling objects, we should use the universal property of $\operatorname{Coker} \theta_A^n$.

Note that

$$(\operatorname{coker} \theta_B^n \circ \beta) \circ \theta_A^n = \operatorname{coker} \theta_B^n \circ (\theta_B^n \circ \alpha) = 0 \circ \alpha = 0.$$

As claimed, there is a unique morphism $H^n(f^\bullet)\colon \operatorname{Coker} \theta_A^n \to \operatorname{Coker} \theta_B^n$.

2.5 H^n is a functor