The category of chords, cluster categories and interleavings

Thomas Brüstle, Christian Chávez*, Emily Cliff

Université de Sherbrooke, QC, Canada

We introduce a k-linear category $\mathbb{X}$ whose objects are two-element subsets of S^1 and whose morphisms are determined by the winding order of their endpoints. We describe its morphisms geometrically, and study its discrete full subcategories $\mathbb{X}_n$ arising from regular n-gons. We then investigate the connection between $\mathbb{X}_n$ and cluster categories of type A, and the interleaving distance on $\mathbb{X}$ given by a monoidal functor and a monoidal weight on S^1.

The winding order on S^1

Let k denote an arbitrary field, fixed throughout. The circle S^1 is regarded as the additive group $\mathbb{R}/2\pi\mathbb{Z}$ and the usual composition of functions is denoted by juxtaposition.

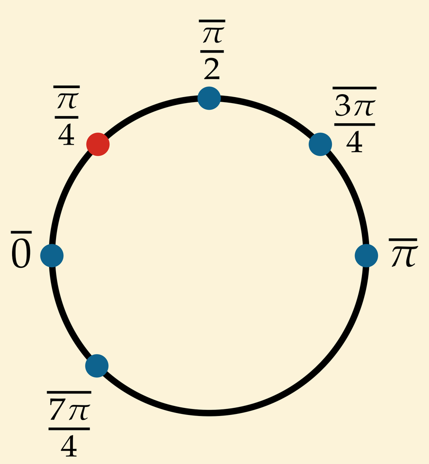
Definition

A finite sequence of points $x_1, \ldots, x_n$ in S^1 is said to be in *winding order* if there exist lifts $\tilde{x}_1 \in x_1, \ldots, \tilde{x}_n \in x_n$ such that

$$\tilde{x}_1 \leq \cdots \leq \tilde{x}_n < \tilde{x}_1 + 2\pi.$$

In such a case we write $x_1 \preccurlyeq \cdots \preccurlyeq x_n$. We write $x_i \prec x_{i+1}$ when $\tilde{x}_i < \tilde{x}_{i+1}$.

Example



Starting at $\frac{\pi}{4}$ and moving clockwise around the circle, we have

$$\tfrac{\pi}{4} \prec \tfrac{\pi}{2} \prec \tfrac{3\pi}{4} \prec \bar{\pi} \prec \tfrac{7\pi}{4} \prec \bar{0}$$

since we have the chain of lifts

$$\tfrac{\pi}{4} < \tfrac{\pi}{2} < \tfrac{3\pi}{4} < \pi < \tfrac{7\pi}{4} < 2\pi < \tfrac{\pi}{4} + 2\pi$$

on $\mathbb{R}$. In contrast, the sequence $\frac{\pi}{4}, \bar{\pi}, \frac{\pi}{2}$ is not in winding order: the first lift of $\frac{\pi}{2}$ after π is $\frac{5\pi}{2}$, but $\frac{5\pi}{2} > \frac{\pi}{4} + 2\pi$.

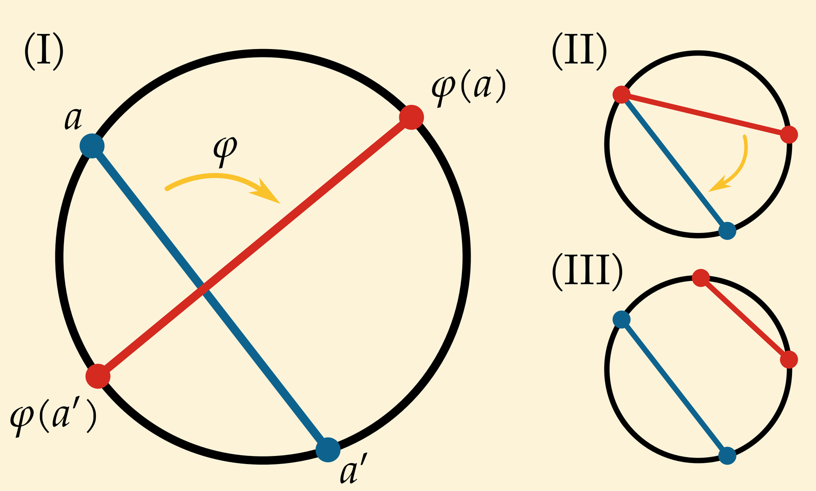
For a set $X \subset S^1$ with exactly two elements, we let σ_X denote the map $X \to X$ that swaps the points of X. We say that a bijection $\varphi\colon A \to B$ between two-element subsets $A, B \subset S^1$ is *admissible* if

$$a \preccurlyeq \varphi(a) \preccurlyeq \sigma_A(a) \preccurlyeq \varphi\sigma_A(a)$$

for some $a \in A$. The set of admissible bijections from A to B is denoted $\mathrm{Adm}(A, B)$. For example, Id_X is admissible whereas σ_X is not.

Using the homeomorphism $\Phi\colon \bar{x} \mapsto (-\cos x, \sin x)$ from S^1 to the unit circle in $\mathbb{R}^2$, we can define the geometric realization of any two-element set $X \subset S^1$ as the convex hull of $\Phi(X)$. This line segment is called the *chord* determined by X, and we also refer to X itself as a chord.

There exists exactly one admissible bijection $A \to B$ if and only if the chords A, B intersect in the unit disk.



The figure shows three situations: in (I) there is an admissible bijection $\varphi\colon \{a, a'\} \to \{\varphi(a), \varphi(a')\}$ because the chords intersect in the interior of the unit disk; if the chords share an endpoint, as in (II), there is also an admissible bijection; if the chords don't intersect, no admissible bijection exists, as in (III).

Theorem

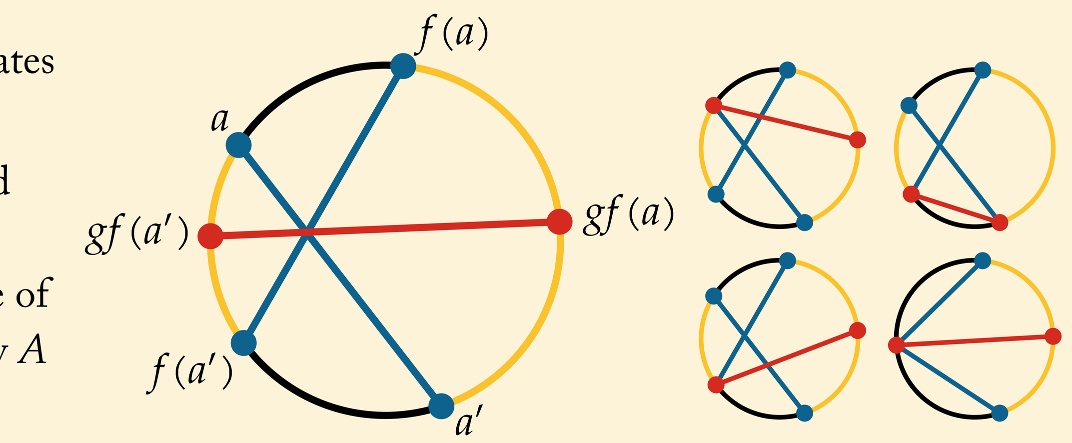
Let $f \in \mathrm{Adm}(A, B)$, $g \in \mathrm{Adm}(B, C)$, and $h \in \mathrm{Adm}(C, D)$. Suppose that hgf is admissible. Then gf is admissible if and only if hg is admissible.

Corollary

For $f \in \mathrm{Adm}(A, B)$ and $g \in \mathrm{Adm}(B, C)$ define $g \circ f = gf$ if gf is admissible and $g \circ f = 0$ otherwise. Then $\circ$ is associative.

The category of chords on S^1

The figure on the right illustrates the fact that the composition $A \xrightarrow{f} B \xrightarrow{g} C$ is nonzero if and only if each endpoint of $C = gf(A)$ lies in exactly one of the yellow arcs determined by A and B.



As a result of the last corollary, we have the following definition.

Definition

The category $\mathbb{X}$ of chords on S^1 consists of the following data:

(i) Objects: two-element subsets of S^1, called chords.

(ii) Morphisms: for two chords A and B, we define $\mathrm{Hom}(A, B)$ as the free k-vector space with basis $\mathrm{Adm}(A, B)$. The identity morphism of A is the admissible bijection Id_A.

(iii) Composition: for two basis morphisms $f \in \mathrm{Adm}(A, B)$ and $g \in \mathrm{Adm}(B, C)$ we define

$$g \circ f = \begin{cases} gf & \text{if } gf \in \mathrm{Adm}(A, C), \\ 0 & \text{else.} \end{cases}$$

We extend $\circ$ bilinearly.

Theorem

(i) For any distinct chords A, B and morphisms $A \xrightarrow{f} B \xrightarrow{g} A$, we have $g \circ f = 0$.

(ii) Every nonzero non-isomorphism factors non-trivially through another chord. Therefore, there are no irreducible morphisms in $\mathbb{X}$.

Remark. The category $\mathbb{X}$ can be embedded as a full subcategory of a new category where the singletons of S^1 are allowed as objects, viewed as *degenerate chords*.

Discrete subcategories of $\mathbb{X}$

Fix a positive integer n and let $V_n = \left\{ \frac{2\pi}{n} i \,\middle|\, i \in \{0, 1, \ldots, n-1\} \right\}$. Let $\mathbb{X}_n$ denote the full subcategory of $\mathbb{X}$ whose objects are two-element subsets of V_n.

Theorem

Let $A, B \in \mathbb{X}_n$ and $f \in \mathrm{Adm}(A, B)$. Then f is irreducible if and only if

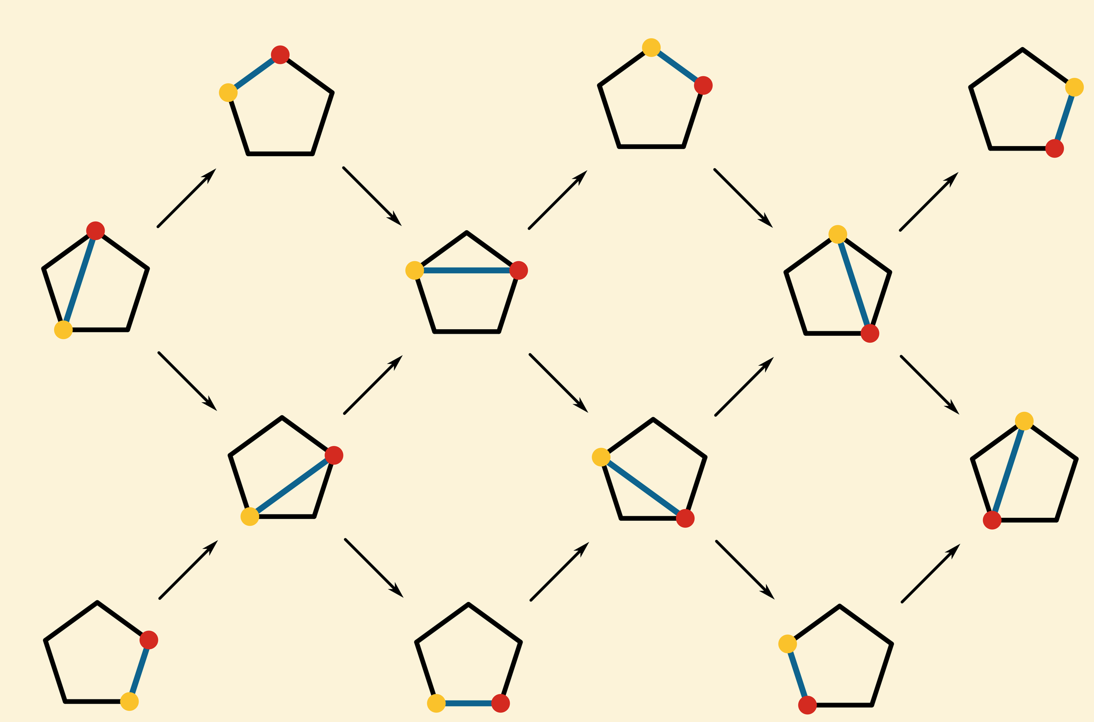
(i) the chords share exactly one point, say $A \cap B = \{p\}$,

(ii) the two points in $A \triangle B$ are adjacent in V_n, and

(iii) f fixes p.

Example

The following is the Auslander-Reiten quiver of $\mathbb{X}_5$. Irreducible morphisms rotate a chord clockwise one endpoint at a time, while keeping the other fixed. Identical chords are identified.



Connections & work in progress

Cluster categories

Caldero, Chapoton, and Schiffler gave a geometric model in [CCS] of the cluster category of type A_n using diagonals of a regular $(n+3)$-gon. We conjecture that our construction also provides a geometric characterisation of the cluster category.

Conjecture

For $n \geq 4$, let I_n be the ideal generated by the identity morphisms of the chords in $\mathbb{X}_n$ that have adjacent endpoints. We conjecture that $\mathrm{add}(\mathbb{X}_n/I_n)$ is equivalent to the cluster category of type A_{n-3}.

We also investigate the relationship between $\mathbb{X}$ and the category $\widetilde{\mathrm{Ind}}\mathcal{C}_\pi$ introduced in [IT].

Interleaving distance and actegories

Let $\mathbb{P}$ denote the category of persistence modules $\mathrm{Fun}((\mathbb{R}, \leq), \mathrm{vect}_k)$. For interval modules, $\mathrm{Hom}_{\mathbb{P}}(\mathbf{k}^{[a,b]}, \mathbf{k}^{[c,d]}) \cong k$ whenever $c \leq a \leq d \leq b$. This motivated us to ask whether chords in S^1 can play the role of interval modules. In this regard, we considered the framework of [MN].

- Let G be a monoidal category with unit e and monoidal product $\otimes$. A *G-actegory* is a pair $(\mathbf{A}, T)$ where $\mathbf{A}$ is a category and T is a monoidal functor $G \to \mathrm{End}(\mathbf{A})$.
- A *monoidal weight* on G is a function $w\colon G_0 \to [0, \infty]$ such that $w(e) = 0$ and $w(g \otimes h) \leq w(g) + w(h)$ for all $g, h \in G_0$.

Definition

Let $T\colon G \to \mathrm{End}(\mathbf{A})$ be a monoidal functor and w a monoidal weight on G.

(i) Two objects $A, B \in \mathbf{A}$ are *(g,h)-interleaved* for $g, h \in G_0$, if there exists a pair of morphisms $\phi\colon A \to T_g(B)$, $\psi\colon B \to T_h(A)$ in $\mathbf{A}$ and a pair of morphisms $\alpha\colon e \to g \otimes h$, $\beta\colon e \to h \otimes g$ in G such that the diagrams

$$\begin{tikzcd} A \arrow[rr, "T(\alpha)_A"] \arrow[dr, "\phi"'] & & T_{g\otimes h}(A) \\ & T_g(B) \arrow[ur, "T_g(\psi)"'] & \end{tikzcd} \qquad \begin{tikzcd} & T_h(A) \arrow[dr, "T_h(\phi)"] & \\ B \arrow[ur, "\psi"] \arrow[rr, "T(\beta)_B"'] & & T_{h\otimes g}(B) \end{tikzcd}$$

commute.

(ii) The associated *G-interleaving distance* is the function $d\colon \mathbf{A}_0 \times \mathbf{A}_0 \to [0, \infty]$ given by

$$d(A, B) = \inf\{\max\{w(g), w(h)\} \mid A \text{ and } B \text{ are } (g,h)\text{-interleaved}\}.$$

We are interested in the interleaving distances on $\mathbb{X}$.

- Regard S^1 as a discrete monoidal category under addition.
- Rotation of chords defines a monoidal functor $\tau\colon S^1 \to \mathrm{End}(\mathbb{X})$. For $\theta \in S^1$, let $\tilde{\theta} \in (-\pi, \pi]$ be its unique representative and define $w(\theta) = |\tilde{\theta}|$.
- The action τ and the weight w determine an interleaving distance on $\mathbb{X}$ that measures the amount of rotation required for two chords to become interleaved.

We aim to investigate other actions and weights that capture finer geometric information.

Contact information

- thomas.brustle@usherbrooke.ca
- christian.chavez@usherbrooke.ca
- emily.cliff@usherbrooke.ca

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[MN] P. K. McFaddin and T. Needham, *Interleaving distances, monoidal actions and 2-categories*, Algebr. Geom. Topol. **26** (2026), no. 1, 227–281.