Homotopy Theory for Finite Categories

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   - Homotopy of Paths
   - The Fundamental Group of a Finite Category
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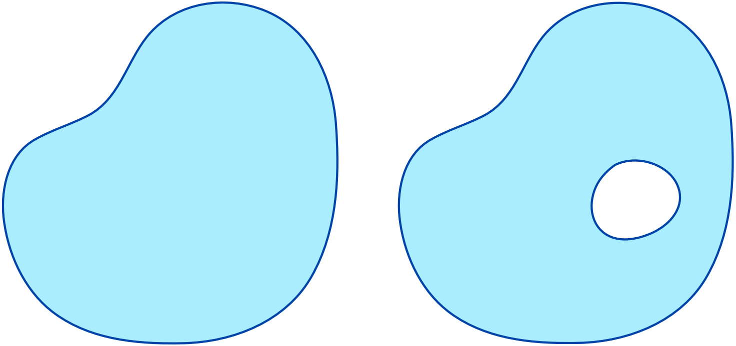
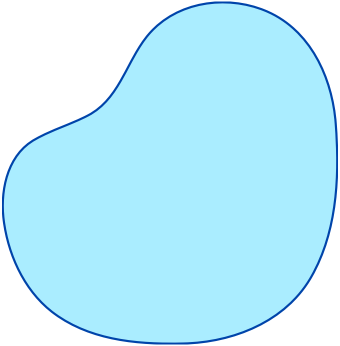
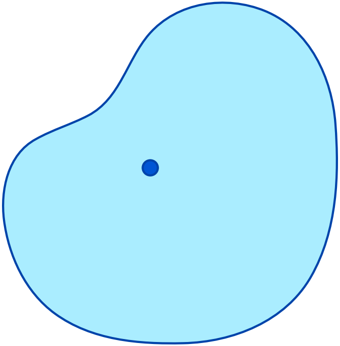
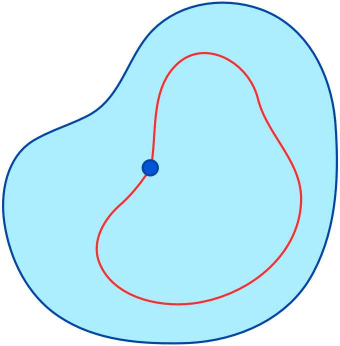
Introduction

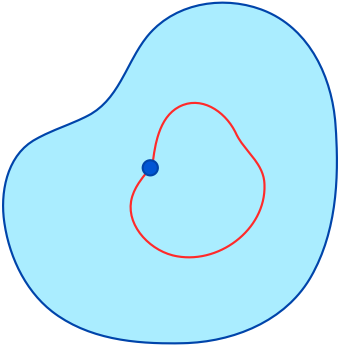
How are these spaces different?

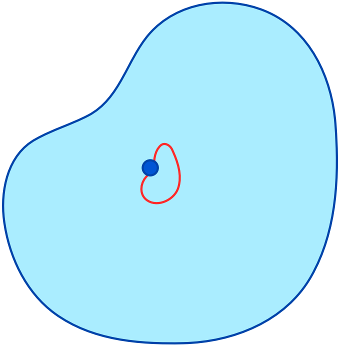


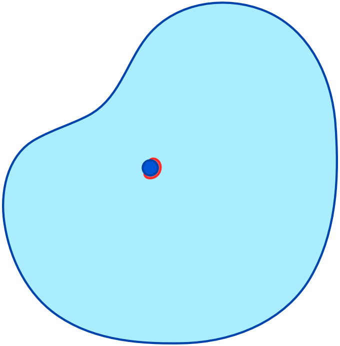


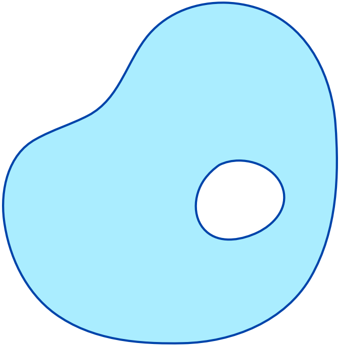


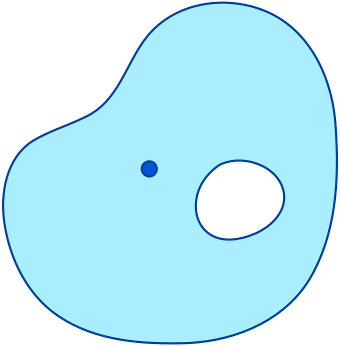


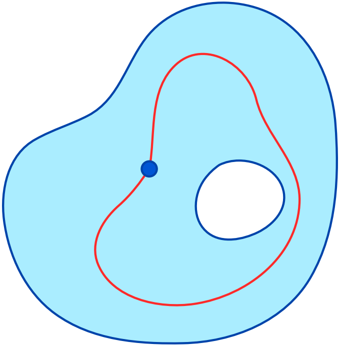


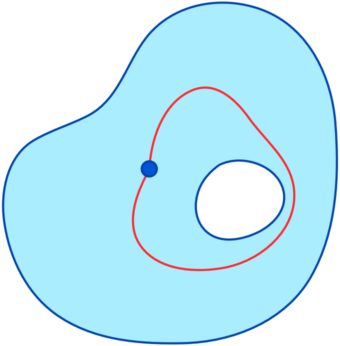


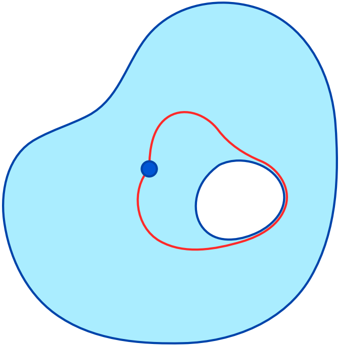


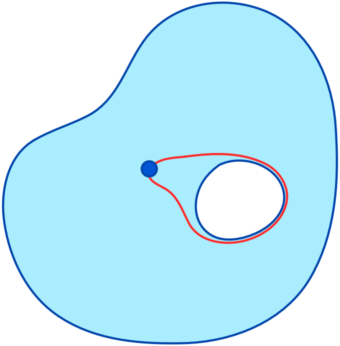




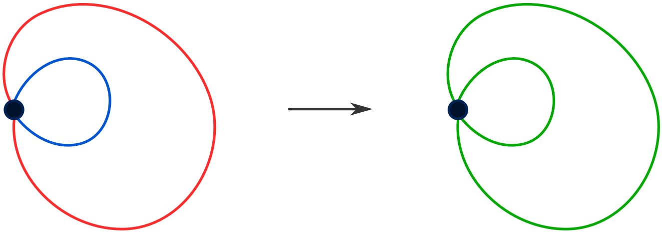








Concatenation



Construction

$$\sim \longrightarrow \{\text{loops}\}/\sim \longrightarrow \left(\{\text{loops}\}/\sim, \text{concatenation}\right)$$

path homotopy — identify loops — π_1

Definition

- A **category** is a collection of objects and morphisms, where the morphisms can be composed associatively and every object has an identity morphism.
- A **finite category** is one whose collection of morphisms is finite.
- A **functor** is an assignment between categories that maps objects to objects and morphisms to morphisms, which preserves identities and compositions.

Some categories: $\mathbf{Set}$, $\mathbf{Top}$, $\mathbf{Grp}$.

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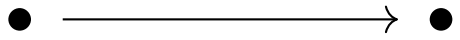


$\mathbf{1}$

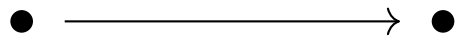
$\mathbb{D}^2$

$\mathbb{S}^1$

$\mathcal{B}K_4$

Some Finite Categories

$\mathbf{0}$



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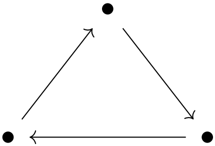
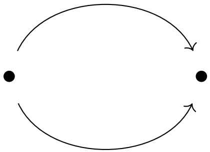
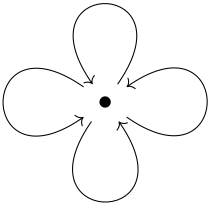


$\mathbf{D}^2$

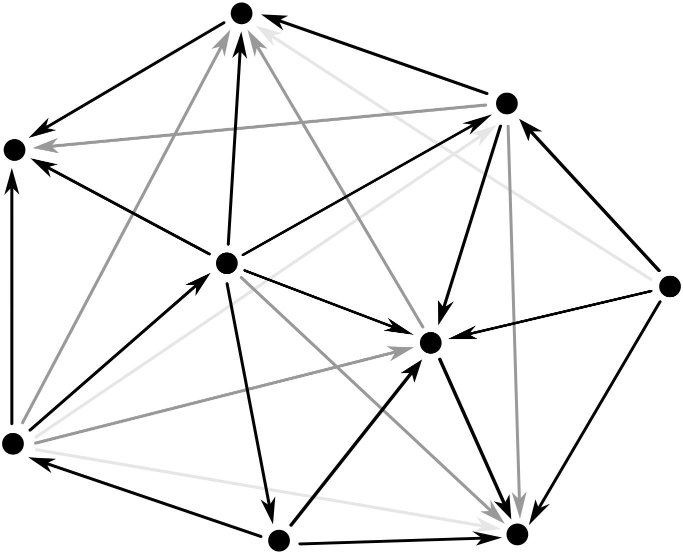


$\mathbf{S}^1$



$\mathcal{B}K_4$

Motivation: “Holes” in a Finite Category



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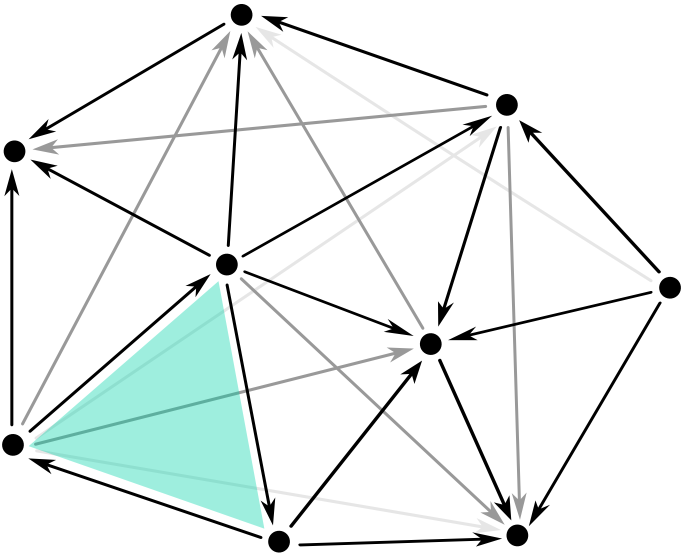

Motivation

- The fundamental group

$$\pi_1 \colon \mathbf{Top}_* \to \mathbf{Grp}$$

is a topological invariant.

- Our objective is to define a functor

$$\kappa_1 \colon \mathbf{Cat}_{\mathbf{Fin}} \to \mathbf{Grp}$$

that preserves many of the properties of π_1. The construction must be entirely algebraic.

The Classifying Space of a Category

The Nerve of a Category and its Realization

$$\mathbf{C} \longrightarrow \mathcal{N}\mathbf{C} \longrightarrow \mathcal{B}\mathbf{C} \longrightarrow \pi_1(\mathcal{B}\mathbf{C})$$

Nerve | Realization | FG

The Nerve of a Category and its Realization

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Nerve — Realization — FG

The classifying spaces of $\mathbf{S}^1$ and $\mathbf{T}^2$

Let $\mathbf{T}^n := \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$. We have

$$\mathcal{B}\mathbf{S}^1 \cong S^1 \quad \text{and} \quad \mathcal{B}\mathbf{T}^n \cong T^n.$$

$$\mathbf{C} \longrightarrow \mathcal{N}\mathbf{C} \longrightarrow \mathcal{B}\mathbf{C} \longrightarrow \pi_1(\mathcal{B}\mathbf{C})$$

Nerve — Realization — FG

This is too much work. Can we get straight to $\mathbf{Grp}$?

$$\mathbf{C} \longrightarrow \kappa_1(\mathbf{C})$$

Yes.

$$\mathbf{C} \longrightarrow \mathcal{N}\mathbf{C} \longrightarrow \mathcal{B}\mathbf{C} \longrightarrow \pi_1(\mathcal{B}\mathbf{C})$$

Nerve — Realization — FG

This is too much work. Can we get straight to $\mathbf{Grp}$?

$$\mathbf{C} \longrightarrow \kappa_1(\mathbf{C})$$

Yes.

Homotopy Theory for Finite Categories

The preordered category $\mathbf{\Lambda}$ is defined by this diagram:

$$\cdots \longleftarrow -2 \longrightarrow -1 \longleftarrow 0 \longrightarrow 1 \longleftarrow 2 \longrightarrow \cdots$$

It will play the role of the unit interval in the classical theory of homotopy.

Let $\mathbf{C}$ be a finite category. A path α in $\mathbf{C}$ is a functor $\mathbf{\Lambda} \to \mathbf{C}$ that induces a finite diagram of the form

$$\alpha(m) \xrightarrow{\alpha(m \to m+1)} \alpha(m+1) \longleftarrow \cdots \longrightarrow \alpha(n-1) \xleftarrow{\alpha(n \to n-1)} \alpha(n)$$

Constant path

$$A \longrightarrow A \longleftarrow A \longrightarrow \cdots \longleftarrow A \longrightarrow A \longleftarrow A$$

Inverse path

$$\alpha : \quad A_0 \longrightarrow A_1 \longleftarrow A_2 \longrightarrow \cdots \longleftarrow A_{n-2} \longrightarrow A_{n-1} \longleftarrow A_n$$

$$\downarrow$$

$$\overline{\alpha} : \quad A_n \longrightarrow A_{n-1} \longleftarrow A_{n-2} \longrightarrow \cdots \longleftarrow A_2 \longrightarrow A_1 \longleftarrow A_0$$

Paths induced by a morphism

$$A \xrightarrow{f} B$$

$$A = A = A \xrightarrow{f} B = B = B$$

Definition (Concatenation of paths)

Let α and β be given by the following diagrams:

$$\alpha(m) \longrightarrow \alpha(m+1) \longleftarrow \cdots \longrightarrow \alpha(n-1) \longleftarrow \alpha(n)$$

$$\beta(p) \longrightarrow \beta(p+1) \longleftarrow \cdots \longrightarrow \beta(q-1) \longleftarrow \beta(q)$$

Their concatenation is given by this diagram:

$$\alpha(m) \longrightarrow \cdots \longleftarrow \alpha(n) = \beta(p) \longrightarrow \cdots \longleftarrow \beta(q)$$

Theorem (Associativity of $\cdot$)

Let α, β, and γ be paths in $\mathbf{C}$. Then

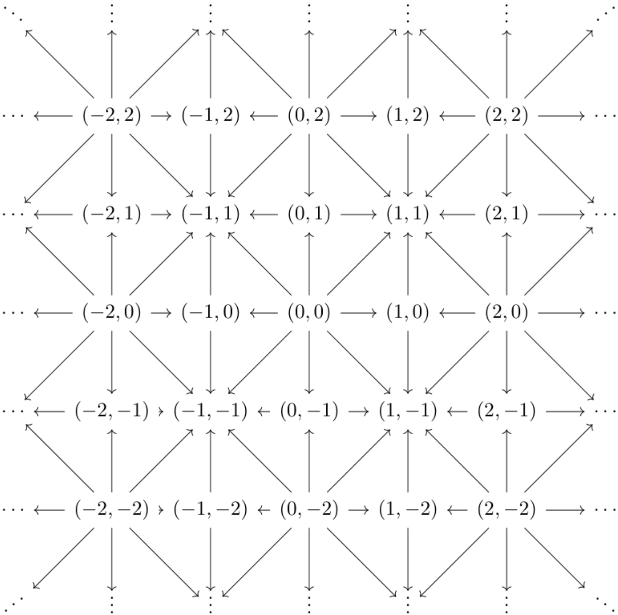
$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

whenever the concatenations are defined.

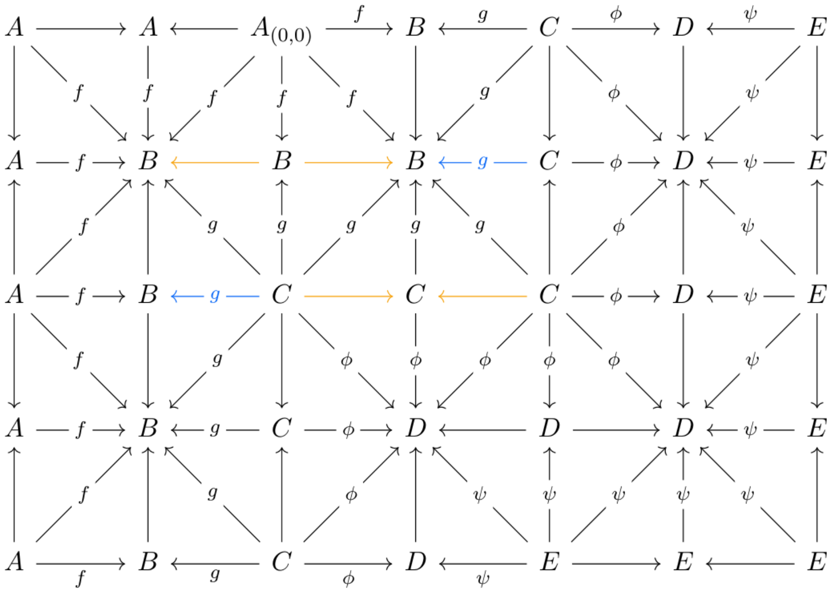
Definition (Homotopy of paths)

Let $\alpha, \beta \in \Phi_{\mathbf{C}}(A, B)$. A **path homotopy** $\mathcal{H}\colon \mathbf{\Lambda} \times \mathbf{\Lambda} \to \mathbf{C}$ from α to β is a commutative diagram between α and β.

Digression: The Product category $\Lambda \times \Lambda$



Translation of Paths



Let $\mathbf{C}$ be a finite category and fix a $\mathbf{C}$-object A.

- The set of loops in $\mathbf{C}$ based at A is denoted $\Omega(\mathbf{C}, A)$.
- We denote

$$\kappa_1(\mathbf{C}, A) = \Omega(\mathbf{C}, A)/\sim$$

- **Objective:** to endow $\kappa_1(\mathbf{C}, A)$ with a group structure

Theorem

Let $\cdot$ be the operation on $\kappa_1(\mathbf{C}, A)$ given by

$$[\,\alpha\,] \cdot [\,\beta\,] = [\,\alpha \cdot \beta\,].$$

Then $\cdot$ is well defined.

Proof.

Follows from the fact that if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, and if $\alpha \cdot \beta$ is defined, then $\alpha' \cdot \beta'$ is defined and

$$\alpha \cdot \beta \sim \alpha' \cdot \beta'.$$

□

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Theorem

The set $\kappa_1(\mathbf{C}, A)$ is a group under the operation of product of path classes of loops based at A.

Proof.

- **Associativity**: by construction, $\cdot$ is associative.
- **Identity**: follows from the fact that $\hat{A} \cdot \alpha \sim \alpha$ and $\alpha \cdot \hat{A} \sim \alpha$.
- **Inverses**: given by inverse paths, which satisfy $\alpha \cdot \overline{\alpha} = \hat{A} = \overline{\alpha} \cdot \alpha$.

□

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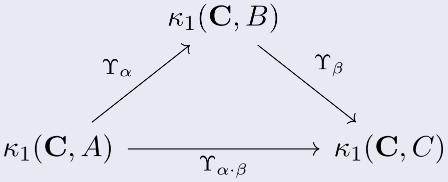
Theorem

Let A and B be two $\mathbf{C}$-objects and let α be a path from A to B. Define

$$\Upsilon_\alpha \colon \kappa_1(\mathbf{C}, A) \to \kappa_1(\mathbf{C}, B) \ : \ [\gamma] \mapsto [\overline{\alpha}] \cdot [\gamma] \cdot [\alpha].$$

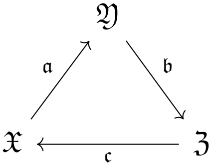
Then Υ_α is a group-isomorphism. Moreover, if β is a path from B to C, the diagram



commutes.

Change of Base Object: D^2

We know $\mathbf{D}^2$ is given by the following diagram:



Since $\mathbf{D}^2$ is **path-connected** and this diagram commutes, we have

$$\kappa_1(\mathbf{D}^2, \mathfrak{X}) \cong \kappa_1(\mathbf{D}^2, \mathfrak{Y}) \cong \kappa_1(\mathbf{D}^2, \mathfrak{Z}) \cong 0.$$

The fundamental group of a product of topological spaces is well behaved under products, up to isomorphism. In our theory, this is also true.

Theorem

For path-connected finite categories $\mathbf{C}$ *and* $\mathbf{D}$*,*

$$\kappa_1(\mathbf{C} \times \mathbf{D}) \cong \kappa_1(\mathbf{C}) \times \kappa_1(\mathbf{D}).$$

We have proved yet another equivalence with that of the classical theory of homotopy of topological spaces.

Theorem

We have

$$\kappa_1(\mathbf{S}^1) \cong \pi_1(S^1) \cong \pi_1(\mathcal{B}\mathbf{S}^1) \cong \mathbb{Z}$$

and

$$\kappa_1(\mathbf{T}^n) \cong \pi_1(T^n) \cong \pi_1(\mathcal{B}\mathbf{T}^n) \cong \mathbb{Z}^n.$$

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